An unknown bequest of the blessed Francesco Faà di Bruno (1825-1888)

"Miracolous analytical solutions for Merton inter-temporal portfolio choice problem"

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Faà di Bruno's life (1825-1888)



Faà di Bruno's earlier life

1841-1853: study at the Royal Military Academy of Turin, with the aim of making a career in the army.

⇛ In 1853 he leaves the army and takes up the study of mathematics.

1853-1861: he travels to Paris where he studies at the Sorbonne under Cauchy who-:

... *"he admired, not only for his genius, but also for his religious fervour and his philanthropy"-*

⇛ At the Sorbonne he was in the same classes as Hermite and Leverrier, who shared in the discovery of the planet Neptune.

⇛ After graduating, he returns to Turin, where he studies for his doctorate, which he obtains in 1861 from both the universities of Paris and Turin.

1871- Professor at the University of Turin, where he is appointed to the Chair of Higher Analysis in 1876.

Faà di Bruno's later life

Return to Turin: Faà di Bruno comes in contact with Giovanni Bosco.

Giovanni Bosco: was ordained a Roman Catholic priest in 1841 in Turin and began to work there, helping boys looking for work in the city.

⇛ He provided boys with education, religious instruction, and recreation, and founded, with others, the Society of St Francis de Sales in 1859.

October 1876: Faà di Bruno is ordained a Roman Catholic priest in Rome.

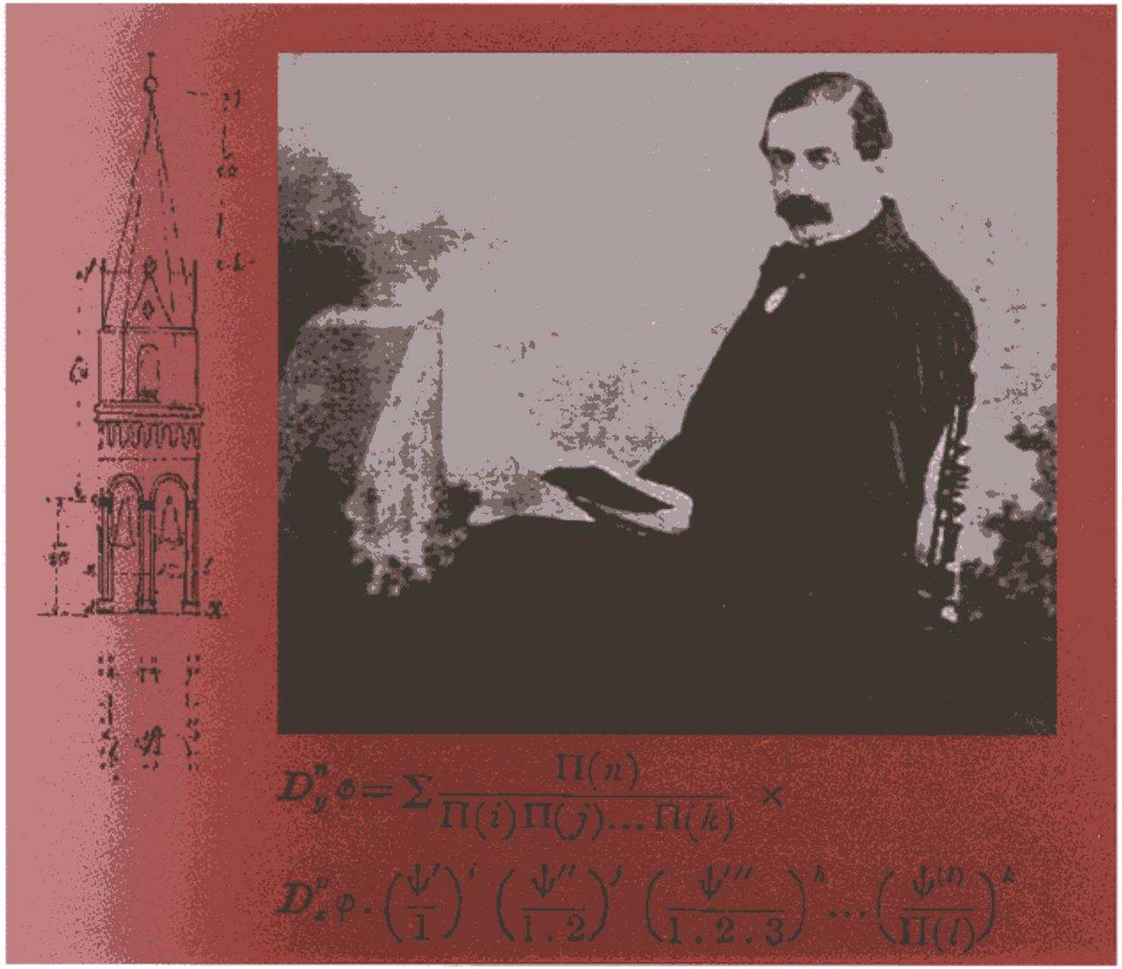
⇛ He founds the religious order "Suore Minime di Nostra Signora del Suffragio" in order to direct and work for girls gathered in a house.

⇛ There, a number of mathematics books were published including one by Faà di Bruno himself on elliptic functions.

1898: The printing press was purchased by Peano for 407 lire, and he printed the Rivista di Matematica on it for several years.

1988: Faà di Bruno was declared a Blessed by John Paul II in St. Peter's Square in Rome, about 50 years after Giovanni Bosco's canonisation.

Faà di Bruno's formula



Faà di Bruno's formula (I)

Setting: For differentiable functions f, g consider the composite function:

$$f(g(\gamma))$$

Problem: how to compute the derivative of order k:

$$f(g(\gamma))^{(k)} := \frac{d^k}{d\gamma^k} f(g(\gamma))$$

Leading example: $f(g(\gamma)) = \exp(g(\gamma))$.

$k = 1$: Easy!

$$f(g(\gamma))' = f'(g(\gamma))g'(\gamma) = \exp(g(\gamma))g'(\gamma)$$

$k = 2$: Easy, but a bit annoying!

$$f(g(\gamma))'' = (\exp(g(\gamma))g'(\gamma))' = \exp(g(\gamma))\left(g''(\gamma) + g'(\gamma)^2\right)$$

$k = 7$: definitely too annoying, it's better to use our time in a different way:

$$f(g(\gamma))^{(k)} = \text{..........?????????????????}$$

Faà di Bruno's formula (II)

Proposition: Given differentiable functions f and g, it follows for an arbitrary derivative of order k:

$$f(g(\gamma))^{(k)} = \sum \frac{n!}{n_1!n_2! \cdot .. \cdot n_k!} f^{(n)}(g(\gamma)) \prod_{j=1}^{k} \left(\frac{g^{(j)}(\gamma)}{j!} \right)^{n_j}$$

where $n = n_1 + n_2 + .. + n_k$ and the sum is over all non negative integers $n_1, .., n_k$ such that $n_1 + 2n_2 + 3n_3 ... + kn_k = k$

Check; $k = 2$**:** The only possible n_1, n_2 are $n_1, n_2 = 0, 1$ and $n_1, n_2 = 2, 0$:

$$f(g(\gamma))^{(2)} = f^{(1)}(g(\gamma))g^{(2)}(\gamma) + f^{(2)}(g(\gamma)) \left(g^{(1)}(\gamma) \right)^2$$

For $f(x) = \exp(x)$, it follows:

$$\exp(g(\gamma))^{(2)} = \exp(g(\gamma)) \left(g^{(2)}(\gamma) + \left(g^{(1)}(\gamma) \right)^2 \right)$$

First miracle: *Faà di Bruno formula works indeed!*

Why higher order derivatives?

Problem: for all (x,γ) and for given functions $\mu(x,\gamma)$ and $\sigma(x,\gamma)$, we might want to find function $g(x,\gamma)$ solving a (differential) equation like:

$$0 = e^{\gamma g(x,\gamma)} + \mu(x,\gamma) g_x(x,\gamma) + \sigma(x,\gamma) g_{xx}(x,\gamma) =: H(x,\gamma,g(x,\gamma))$$

Assumption: For $\gamma = 0$ solution $g(x,0)$ is known.

Power series approach: Write $H(x,\gamma,g)$ as a power series about $\gamma = 0$:

$$H(x,\gamma,g) = H(x,0,g) + \gamma H_\gamma(x,0,g) + \frac{\gamma^2}{2} H_{\gamma\gamma}(x,0,g) + \ldots$$

and guess a solution of the form

$$g(x,\gamma) = g(x,0) + \gamma g_\gamma(x,0) + \frac{\gamma^2}{2} g_{\gamma\gamma}(x,0) + \ldots$$

$\Rrightarrow$ Collect all terms of the same power in γ and solve term by term to determine g_γ, $g_{\gamma\gamma}, \ldots$!

Key point: to apply this procedure we need to compute an arbitrary $\gamma-$derivative of $e^{g(\gamma,x)}$!

$\Rrightarrow$ Faà di Bruno's formula is perfect to achieve this goal!

Merton problem

Robert C. Merton receiving his Prize from the hands of His Majesty the King.

Merton problem (I)

Optimization problem: Given utility function U of consumption C, where

$$U(C) = \frac{C^{\gamma} - 1}{\gamma}$$

compute portfolio weights and consumption policy $\{\theta_t, C_t\}$ to solve

$$J(X_t, W_t) = \sup_{\{\theta_t, C_t\}} E_t \left[\int_t^{\infty} e^{-\delta(s-t)} U(C_s) ds \right] \quad ; \quad \delta > 0$$

subjet to dynamic budget constraint for wealth process $\{W_t\}$:

$$\begin{aligned} dW_t &= W_t \left[r(X_t) + \theta_t(\mu(X_t) - r(X_t)) - \frac{C_t}{W_t} \right] dt + W_t \theta_t \sigma(X_t) dZ_t \\ dX_t &= \mu_X(X_t) dt + \sigma_X(X_t) dZ_t^X \end{aligned}$$

where $r, \mu, \sigma, \mu_X, \sigma_X$ are given functions and (Z, Z^X) is a bivariate Brownian motion process with correlation ρ.

Merton problem (II)

Bellman equation: using the transformation

$$J(X,W) = \frac{1}{\delta}\frac{\left(e^{g(X)}W\right)^{\gamma} - 1}{\gamma}$$

solving Merton problem is equivalent to solving the differential equation:

$$\begin{aligned} 0 &= r + \frac{1}{\gamma}\left((1-\gamma)\left(\frac{e^{\gamma g}}{\delta}\right)^{1/(\gamma-1)} - \delta\right) + \mu_X g_X \\ &\quad + \frac{1}{2}\frac{1}{1-\gamma}\left(\frac{\mu - r}{\sigma} + \gamma\rho\sigma_X g_X\right)^2 + \frac{\sigma_X^2}{2}(g_{XX} + \gamma g_X^2) \\ &= H(\gamma, g) \end{aligned}$$

Remark: The solution for $\gamma = 0$, i.e. $U(C) = \log C$, is known:

$$C(W_t) = \delta W_t \quad , \quad \theta(X_t) = \frac{1}{1-\gamma}\frac{\mu(X_t) - r(X_t)}{\sigma(X_t)^2}$$

Merton problem and Faà di Bruno's formula

Starting point: following Ferretti and Trojani (2004), we can apply a power series approach to solve the Bellman equation:

$$0 = H(\gamma, g)$$

Power series approach: Write $H(\gamma, g)$ as a power series about $\gamma = 0$:

$$H(\gamma, g) = H(0, g) + \gamma H_\gamma(0, g) + \frac{\gamma^2}{2} H_{\gamma\gamma}(0, g) + \ldots$$

and guess a solution of the form

$$g(x, \gamma) = g(x, 0) + \gamma g_\gamma(x, 0) + \frac{\gamma^2}{2} g_{\gamma\gamma}(x, 0) + \ldots \tag{1}$$

$\Rrightarrow$ Collect all terms of the same power in γ and solve term by term to determine g_γ, $g_{\gamma\gamma}$,...!

Faà di Bruno's formula: it makes possible to write as a power series the exponential term in $H(\gamma, g)$!

Second miracle: *we can characterize completely any term in power series (1) and solve Merton's problem in full generality.*

Miracolous solutions: an example (I)

Setting: r, σ are constant and

$$\mu = r + \sigma X$$

where X follows a mean reverting Ornstein Uhlembeck process. That is,

$$\mu_X = \lambda(\theta - X)\ ,\ \sigma_X = \xi \quad ; \quad \lambda, \xi > 0, \theta \in \mathbb{R}$$

$\Rrightarrow$ The model allows for mean reverting risk premia $\mu - r$!

Complete market setting: For the very special case $\rho = \pm 1$, closed form solutions are known (Wachter (2001)).

$\Rrightarrow$ In the general incomplete market case $\rho \neq \pm 1$ no solution is known!

$\Rrightarrow$ We provide solutions for the latter case using our power series approach!

Miracolous solutions: an example (II)

Setting: To convince you that these are really miracolous solutions, just consider the simplest first order term in the power series for g:

$$g_\gamma(X) = \alpha_0 + \alpha_1 X_2 + \alpha_2 X_2^2 + \alpha_3 X_2^3 + \alpha_4 X_2^4$$

where, for constants $\alpha_{0,i}$, $i = 0, 4$, dependent on the known solution for $\gamma = 0$:

$$\begin{aligned}
\alpha_4 &= \frac{\delta \alpha_{0,4}^2}{2(\delta + 4\lambda)} \\
\alpha_3 &= \frac{1}{\delta + 3\lambda}\Big(\alpha_{0,3}\alpha_{0,4}\delta + 4\alpha_4\lambda\vartheta\Big) \\
\alpha_2 &= \frac{1}{\delta + 2\lambda}\Bigg[\frac{1}{2} + \frac{1}{2}\alpha_{0,3}^2\delta + \alpha_{0,0}\alpha_{0,4}\delta + 2\alpha_{0,4}\rho\sigma + 2\alpha_{0,4}^2\sigma^2 + 6\alpha_4\sigma^2 + 3\alpha_3\lambda\vartheta \\
&\quad -\alpha_{0,4}\rho\log\delta\Bigg] \\
\alpha_1 &= \frac{1}{\delta + \lambda}\Big[\alpha_{0,0}\alpha_{0,3}\delta + \alpha_{0,3}\rho\sigma + 2\alpha_{0,3}\alpha_{0,4}\sigma^2 + 3\alpha_3\sigma^2 + 2\alpha_2\lambda\vartheta - \alpha_{0,3}\rho\log\delta\Big] \\
\alpha_0 &= \frac{1}{\delta}\left[\frac{1}{2}\alpha_{0,0}^2\delta + \frac{1}{2}\alpha_{0,3}^2\sigma^2 + \alpha_2\sigma^2 + \alpha_1\lambda\vartheta - \alpha_{0,0}\rho\log\delta + \frac{1}{2}\delta(\log\delta)^2\right]
\end{aligned}$$

Miracolous solutions: illustration (I)

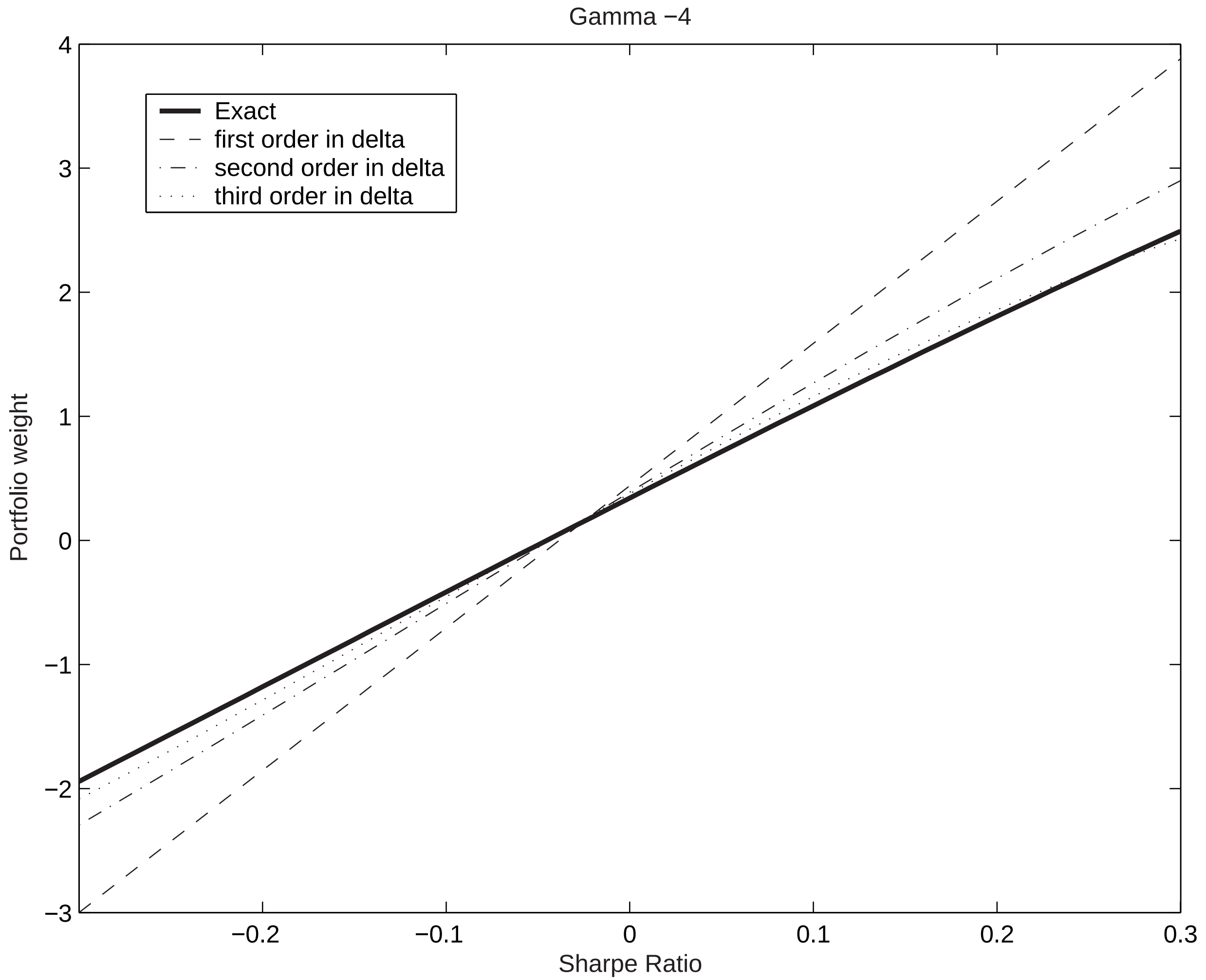


Figure 1: First, second and third order portfolio policies (dashed, dashed-dotted and dotted curves, respectively) as functions of $X \in [-0.3, 0.3]$ and for $\gamma = -4$, $\rho = -1$.

Miracolous solutions: illustration (II)

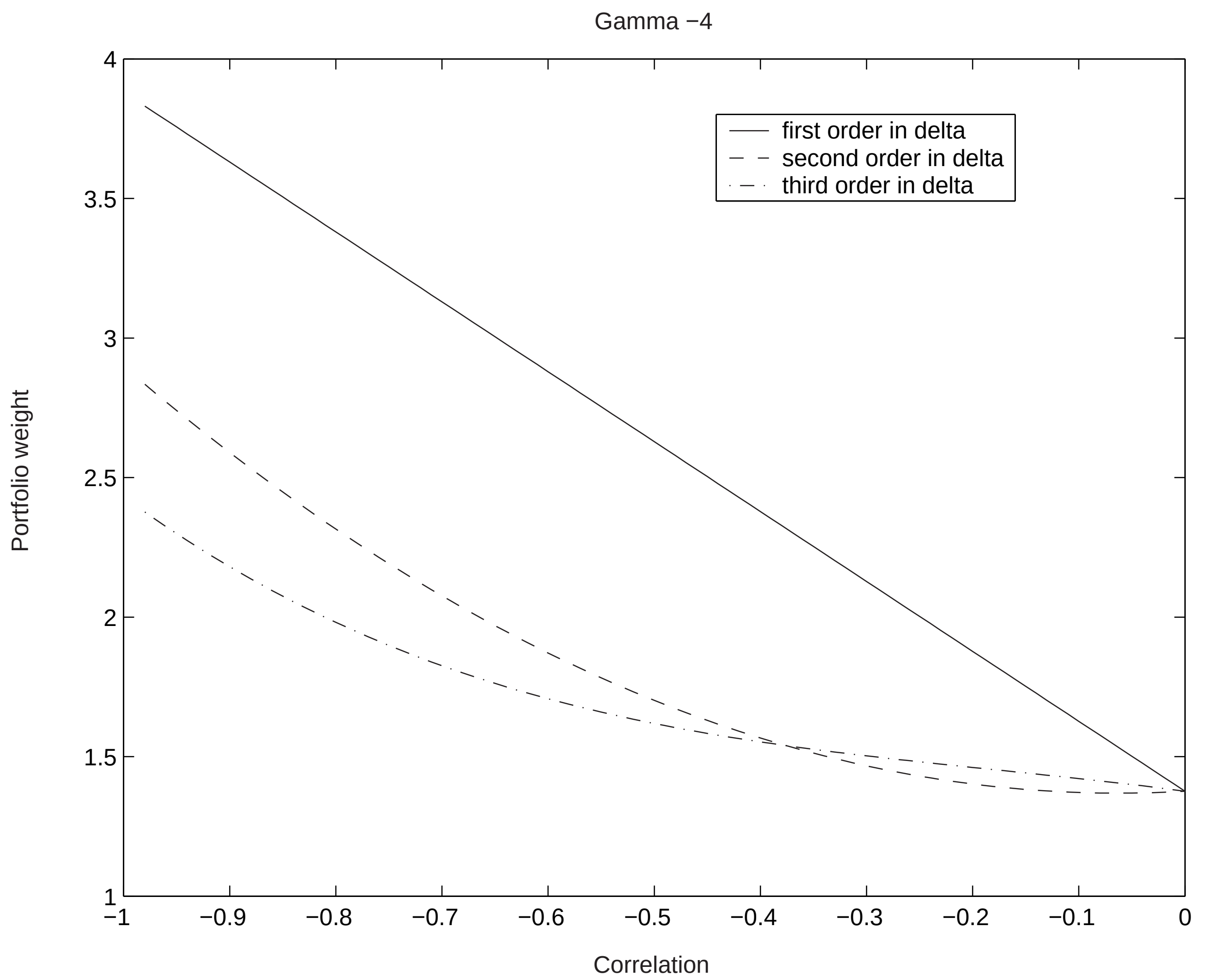


Figure 2: First, second and third order portfolio policies (straight, dashed and dashed-dotted curves, respectively) as functions of $\rho \in [-1, 0]$ and for $\gamma = -4$. All plots are for $X = 0.3$.

Summary

Summary: we could suggest at least two "miracles" by the blessed Francesco Faà di Bruno.

$\Rrightarrow$ As you know, three miracles are necessary, in order to become a saint..........