Script

Solving Economics and Finance problems with MATLAB

Peter H. Gruber

July 29, 2016

Useful shortcuts

+ Stop MATLAB: press `ctrl-c`
+ Restore the MATLAB screen:
  HOME/Layout/Default
  Desktop/Desktop Layout/Default (older MATLAB)
+ List of all symbols: see appendix D.1
+ Getting **help**: see section 2.4
+ **Quick start**: see section 2.3

About the author: Peter H. Gruber has been teaching MATLAB classes since 2005 at the Universities of St. Gallen, Geneva and Lugano. He holds a PhD in Finance from the Università della Svizzera italiana in Lugano, an M.A. in Quantitative Finance and Economics from St. Gallen University and a PhD in physics from the Vienna University of Technology. He has worked for several years at CERN in the field of neutrino physics and is currently a postdoctoral research associate at the Università della Svizzera Italiana in Lugano, Switzerland. His research interests include asset pricing with a focus on volatility risk, high-performance numerical methods and big data.

Contents

1. Introduction

I hear and I forget
I see and I remember
I do and I understand
Confucius

1.1. A problem that cannot be solved analytically

Example 1. One of the most widely used distributions in economics is the standard normal distribution. The probability density function (p.d.f.) is:

$$f(x) = \frac{1}{\sqrt{2\pi}} e^{-\frac{x^2}{2}} \tag{1.1}$$

What is the probability that one draw of a standard normal distributed random variable is below, say, -1? The answer is given by the cumulative distribution function (c.d.f.):

$$Pr(X \leq -1) = F(-1) = \int_{-\infty}^{-1} f(x)dx = \int_{-\infty}^{-1} \frac{1}{\sqrt{2\pi}} e^{-\frac{x^2}{2}} dx \tag{1.2}$$

For the solution of this integral, we usually use a table or the computer. But how is the value of the normal c.d.f. calculated? There is no simple representation. The only thing we can do is to numerically integrate $f(x)$: slice it into small strips and add up their area. Fortunately, computers are very good at such boring, repetitive tasks (see section 11 for the details). This short example illustrates that there are fairly simple problems that can only be solved numerically.

1.2. Numerical methods in economics and finance

Numerical methods are widely used in economics and finance. They are at the core of every econometric estimation. They are used for forecasting, asset pricing, scenario generation, risk management and to optimize everything from truck routes to individual spending behaviour, public policies and portfolios. As a rule of thumb, the more complex or realistic a model becomes, the greater the chance that there is no analytical solution. Literacy in numerical methods is nowadays often a job requirement in both economics and finance.

1.3. About this course

Philosophy

This script is not intended to be a cookbook, but rather a guide on how to invent new recipes. The idea is to give the student a set of coloured stones from which she can make her own mosaic. This means that equal emphasis is laid on understanding and using the numerical methods presented here. This also implies that this script is organized by methods rather than by topics. Techniques useful for, say, option pricing, can therefore be found in the chapters on simulation, nonlinear functions and numerical integration.

The script is organized in three main parts: one for MATLAB, one for numerical methods and one for programming.

Part 1: Introduction to MATLAB

Section 2 is a jump-start into MATLAB and its language with an interactive introduction that employs learning-by-doing starting on page 12. The basic elements of the MATLAB language such as variables, operators, functions and programs are presented in section 3, while the more advanced concepts of flow control (if-then-else, loops), relational and logical operators are covered in section 4.

Part 2: Numerical methods

Section 5 builds the foundation for every empirical work: how to organize and treat data. Section 6 is about linear algebra and its applications to econometrics. Section 7 covers simulation based methods including Monte Carlo simulations. Section 8 discusses nonlinear functions and a few standard problems in numerical mathematics: root finding, fixed points and inverse functions. Section 9 is devoted to numerical differentiation and the associated precision problems. Section 10 covers the important topic of optimization. Sections 11 discusses numerical integration, including a presentation of the FFT-Algorithm (Fast Fourier Transformation).

Part 3: The art of programming

Sections 12 and 13 are two of the most important sections on the road towards a successful numeric project. They develop a framework for the successful management of a programming project and give clear guidelines on how to avoid and detect errors. Sections 14 and 15 contain useful additions: an introduction to parallel computing and an introduction to databases.

The appendices feature reference and review material, and practitioner's tips.

1.4. Prerequisites

Student's knowlegde. No programming knowledge is needed for this script, which aims at being mostly self-contained. Students profit from bachelor level knowledge in mathematics (matrix calculus, functional analysis, complex analysis), finance (definition of basic assets and their payoffs), econometrics (OLS) and probability (moment generating functions).

Software. MATLAB comes with literally hundreds of toolboxes, each of which has to be paid for separately. Therefore this course is limited to using the statistics and econometrics toolboxes, which come with the student edition of MATLAB. Several chapters profit from additional free toolboxes, see page 190 on how to install these toolboxes. All code has been tested with MATLAB 2012b, but should run with any version after 2005b.

1.5. Using this script

The main idea of this course is to combine systematic knowledge with learning by doing. It is best to keep an open MATLAB session while studying.

Theory stands at the beginning of every chapter: to explore how and why certain methods work, as well as their limits and precision.

Examples denoted `|MATLAB examples` illustrate the new concepts. They should be entered into the MATLAB command window while reading the text.

PC labs give the reader a guided tour through a new concept. They are an integral part of this course – some details are only shown in the PC labs. Usage is simple: enter the commands in the command window, watch the results and read the explanation.

Exercises Programming is a matter of experience. The PC labs can only be a starting point, thus the reader is more than invited to try at least a few exercises per chapter. Some of the exercises are used as problem sets for grading, therefore no solutions will be published. Appendix E.1 contains my guidelines for solutions.

Notation and fonts

Capital letters (A) denote matrices, small caps (a) vectors and scalars. Bold small caps ($\mathbf{x}$) always denote vectors, Greek letters ($\alpha, \dots$) always scalars. Vectors are usually column vectors. They are often written using the transpose to save space: $\mathbf{x} = (x, y)' = \left(\begin{smallmatrix} x \\ y \end{smallmatrix} \right)$.

MATLAB commands are printed in the font teletype (e.g. `plot`). To avoid duplication, they are usually not explained in all detail. This is much better done by MATLAB's help function (e.g. `help plot`) or the Mathworks website. Menu items are printed in sans serif, with sub-menus separated by a forward slash (e.g. Help/Full Product Family Help).

NOTE: important remarks are denoted with this symbol.

TIP: practitioner's tips (not mandatory, but useful) are marked with this symbol.

$*$ material or exercises which are a bit more challenging are marked with an asterisk.

^{M} material very specific to the MATLAB language is marked with a superscript M. Students focusing on numerical methods in general may wish to skip these sections.

Resources

This script comes with a set of sample programs, data sets and a set of slides. Solutions to the exercises will not be provided.

Acknowledgements

This text has been vastly improved thanks to input from my students at the University of St. Gallen. Further suggestions and corrections to `peter.gruber@usi.ch` are always welcome. I am grateful to Roman Meyer, Daniel Kohler, Nick Galli and Marco Hürner who gave valuable comments to the first versions of this script. I take the sole responsibility for all remaining errors.

2. A quick start to MATLAB

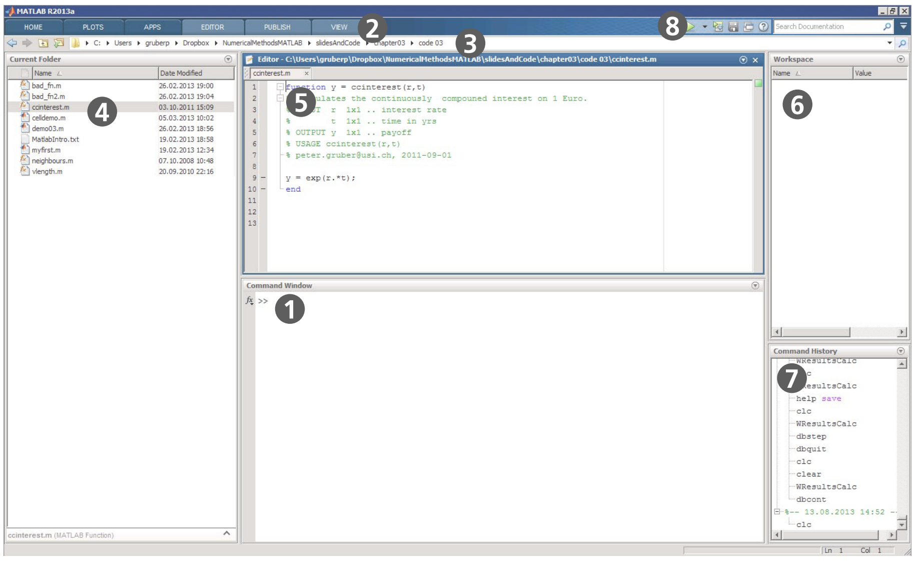
References: Getting started with MATLAB, sections 1, 2 and 6-1 to 6-13; Moler; Hanselman and Littlefield

2.1. Getting started

TIP: If your screen does not look like Fig. 2.1, click Desktop/Desktop Layout/Default.

(1) Command Window. The command window has two uses. First, MATLAB commands can be directly entered here ("*interactive mode*"). This can be used for pocket calculator-like operations. All output – be it from a direct command or from a program – is also displayed in the command window.

Direct input in the command window is a fast way to test new ideas. Once the results are satisfactory, the command history (3) can be used to assemble a program (see Sec. 3.5).



MATLAB R2013a
HOME
PLOTS
APPS
EDITOR
PUBLISH
VIEW
Search Documentation
C: ▸ Users ▸ gruberp ▸ Dropbox ▸ NumericalMethodsMATLAB ▸ slidesAndCode ▸ chapter03 ▸ code 03
Current Folder
Editor - C:\Users\gruberp\Dropbox\NumericalMethodsMATLAB\slidesAndCode\chapter03\code 03\ccinterest.m
ccinterest.m
Workspace
Command Window
Command History
ccinterest.m (MATLAB Function)
Ln 1 Col 1
1
2
3
4
5
6
7
8

Name	Date Modified
bad_fn.m	26.02.2013 19:00
bad_fn2.m	26.02.2013 19:04
ccinterest.m	03.10.2011 15:09
celldemo.m	05.03.2013 10:02
demo03.m	26.02.2013 18:56
MatlabIntro.txt	19.02.2013 18:58
myfirst.m	19.02.2013 12:34
neighbours.m	07.10.2008 10:48
vlength.m	20.09.2010 22:16

```
function y = ccinterest(r,t)
  ulates the continuously  compouned interest on 1 Euro.
%   UT  r  1x1 .. interest rate
%        t  1x1 .. time in yrs
% OUTPUT y  1x1 .. payoff
% USAGE ccinterest(r,t)
% peter.gruber@usi.ch, 2011-09-01

y = exp(r.*t);
end
```

```
fx >>
```

Name	Value

```
WResultsCalc
 c
 esultsCalc
help save
clc
WResultsCalc
dbstep
dbquit
clc
clear
WResultsCalc
dbcont
%-- 13.08.2013 14:52 --
clc
```

Figure 2.1.: MATLAB screen (version 2013a).(1) Command window, (2) MATLAB Toolstrip, (3) Current Directory Bar, (4) Current Directory Window, (5) Program Editor, (6) Workspace, (7) Command History, (8) Quick Access Toolbar.

(2) MATLAB Toolstrip. The toolstrip is organized by topic. Its contents depend on the context. Some important commands can only be found here, e.g. for the editor (indent, outdent, (un)comment) or for graphics.

(3) Current Directory Bar Very important: By default, everything will be loaded or saved from/to this directory.

(4) Current Directory Window The contents of the current directory are shown in this window. If you double-click a program, it will be opened in the program editor (4). If you double-click on a data file, it will be loaded/imported.

(5) Program Editor. Here you can write and edit your programs.

(6) Workspace Note that at the lower end of the current directory window, there is a tab where you can switch between the current directory view and the workspace (2). The workspace is a view into the "memory" of MATLAB. Here, you can see all current variables and their values. When going through the interactive introduction to MATLAB in section 2.3, keep an eye on the workspace and observe how the variables evolve.

(7) Command History. A list of all commands that have been entered, in reverse chronological order. The command history can be used to create a program from trials in the interactive mode (see section 3.5).

(8) Quick Access toolbar. A list of symbols that allow quick access to often-used commands in every context. See the appendix on how to modify the toolbar.

Getting started with MATLAB alternatives. Octave is an open source (=free) program with a command set almost identical to MATLAB. You can think of the Octave user interface as having only a command window. Octave uses special commands for all other elements. The workspace is replaced by the `who` command, the current directory window by the commands `ls, pwd` and `cd`, the command history by `history`. Octave has no program editor, you have to use the external editor of your choice. See Appendix E.1 for how to install Octave and other MATLAB alternatives.

TIP: For a trial of Octave without installing anything on your computer, go to `http://lavica.fesb.hr/octave/octave-on-line_en.php` or `http://www.compileonline.com/execute_matlab_online.php`

2.2. The main elements of the MATLAB language

Variables. Variables are created *implicitly*, i.e. by assigning a value to them. The command `a=1` performs two things: (1) a variable called `a` is created, and (2) the value 1 is assigned to it. Variables can be of different *types*: scalar, vector, matrix (like their mathematical counterparts) and string (for text). Note that the names of variables are case sensitive (so `A` is not the same as `a`). Variables should be named with self-explaining names, see section 3.1.

Arithmetic operators. An operator is best explained with an example: $+, -, *, /$ are all *arithmetic operators*. The hat `^` stands for the power operator, e.g. 5^2 is written as `5^2`. (Remember that $\sqrt[n]{x} = x^{1/n}$). The order in which operators are executed is called *operator precedence*. *Example:* `2+3*5`. The multiplication is executed before the sum. For details, see section 3.2 and Tab. 3.3.

Relational operators. So-called *relational operators* compare two variables and give the results `true` or `false`. Among these are: equal (`==`), not equal (`~=`), less (`<`), less or equal (`<=`), greater (`>`), greater or equal (`>=`). Relations can only be true or false. MATLAB assigns numerical values to these two: `true=1` and `false=0`. *Example:* `2<3` is `true` (or `1`), `4.4==4.5` is `false` (or `0`). For details, see section 4.2.

Logical operators. Most noteworthy are "and"(`&`), "or" (`|`) and not (`~`). They are used to combine two conditions. *Example:* `(2<3) | (4.4==4.5)` is `true`, because one of the two comparisons is true. For details, see section 4.4.

Functions. A MATLAB function works like a standard mathematical function: it takes one or more arguments and produces one or several results. Widely used built-in functions are `log(), exp(), abs(), sqrt()`. MATLAB extends the concept of a function to vectors. *Example:* given a vector $\mathbf{x} = (x_1, x_2, x_3)$, MATLAB computes $\log(\mathbf{x}) = (\log(x_1), \log(x_2), \log(x_3))$.

Apart from these built-in functions, users have the flexibility to write self-defined functions. Note that these work only as well as they have been programmed. User-defined functions are only vector compatible, if the user has explicitly provided for this. Generally, a user-defined function must be saved in a separate m-file. See section 3.4.

Flow control. Many computational algorithms require repeating certain steps, often as a function of previous results. In MATLAB, a block of code can be executed several times (using `for, while`) or subject to a condition (using `if-else, switch-case`). See section 4.3.

Comments. MATLAB ignores all text after a percent sign (`%`) until the end of the line. Start a line with % for long comments or add a short remark after a command in the same line. Comments are an important element of programming style, see Section 12.1.

Data input and output. Calculations would make no sense if there is no output of the results. By default, MATLAB prints the result of *every* calculation in the command window. To suppress the output of intermediate results, end the respective line with a semicolon (`;`).

It is not customary to let the user interactively input data. Small amounts of data such as parameter choices are often specified in the first block of a program. Larger amounts of data are usually stored in a separate data file. MATLAB can directly access many database systems and read and write various file formats, including CSV and MS Excel. See Chapter 5.

Graphics are a special form of data output. MATLAB supports a wide range of plot types, see Appendix D.2.

Toolboxes are libraries of useful functions for a certain field. The Mathworks sell some 90 different toolboxes, of which about a dozen are relevant for economics and finance. Programs that rely on a toolbox will only run on machines where this toolbox is installed; this is impractical if several people work on a project. There is an increasing number of open source toolboxes, see the web links in section 2.5.

2.3. PC-Lab: An interactive introduction to MATLAB

MATLAB is best explained by using it. Just enter the `commands` in the command window (see Fig. 2.1) and press return. Observe the output and possible changes in the workspace. A short explanation is given for every command; detailed explanations follow in the two subsequent chapters.

Before getting started

TIP: MATLAB's `diary` command makes it possible to save the results of this interactive introduction. First, verify that the **current directory** (see Fig. 2.1) is set suitably.

TIP: If you have not yet done so, bring the workspace to the front and keep an eye on the MATLAB workspace while working through this introduction.

`diary('intro.txt')`	Set the diary file name and start recording.
`% My first MATLAB session`	Start the line with a '%' to make a comment. They will show up in the diary and help you remember.

Matlab as a pocket calculator

`1+1`	
`2+3*5`	MATLAB observes the correct *operator precedence.*

Variables

`a`	Error message, because the variable `a` has not yet been created.
`a=2`	This does two things: (1) create a variable `a` and (2) assign the value 2 to this variable. Watch the workspace (see Fig. 2.1).
`a`	Recall the value of the variable.
`b=3.5`	Define a variable b with value 3.5 (its a dot and not a comma!)
`b=3,5`	The comma (,) means next command ...
`b=3, c=5`	e.g. to write two commands in one line (rarely used).
`A`	You receive an error message; there is no variable `A` (only small `a`). Commands and the names of variables and files are *case sensitive.*

Arithmetic operators

`a+b`	Arithmetic operators are `+ - / *` and `^` for "power".
`a-b`	Negative numbers are simply entered with a minus in front.
`d=a+b`	Assign the result of a calculation to a new variable.
`a+b*c`	Operator precedence
`(a+b)*c`	Round brackets have the highest precedence. Note: every shape of brackets has its own meaning, see Appendix D.1.
`((a+b)*c+a)/b`	Several levels of brackets are all defined with the round bracket ().
`a^b`	Power.
`a^b+c`	This may produce an undesired result ...
`a^(b+c)`	If you want to enter a^{b+c}, you have to use a bracket.
`a^2`	Squared
`sqrt(a)`	Square root. But is there also a command for the third root of a?
`a^(1/3)`	There is none. We need to use our mathematics knowledge: $\sqrt[3]{x} = x^{1/3}$.
`a^0.33`	Compare the resuts. $\boxed{\text{TIP:}}$ never perform computations for the computer!

Some floating point formatting

`format long`	Display output with full 15 digit precision.
`[2xcursor up]`	Cursor up gives the last input. Cursor up twice should give you `a^0.33`.
`[0.01 1250 2.25]`	Typical financial data.
`format bank`	Useful.
`format short`	NOTE: `format` only changes the display – not the result of the calculation.

Functions

`log(a)`	Natural logarithm (some people write ln() for it) ...
`log10(a)`	... as opposed to the logarithm with basis 10.
`exp(a)`	Exponential.
`abs(-3)`	Absolute value.
`rand`	Function without arguments. Try several times. Use `[cursor-up]`
`help rand`	Does rand really have no arguments? Use the help function to find out.

Errors and NaN (not a number)

`log(0)`	A pocket calculator would give an error, here.
`e=0/0`	Another error. `NaN` stands for *not a number*.
`1+e`	Calculations with `NaN` are possible, but most of the time not very useful.

Vectors

`xRow=[1, 3, 7]`	This creates a row vector: commas separate the columns of a vector.
`yRow=[2 5 1]`	The commas are not needed: use spaces as shortcut.
`aCol=[3; -1; 1]`	The semicolon means "new line" (We mostly use column vectors.)
`bCol=[3 -1 1].'`	Transposing a row vector is a fast way to enter a column vector.
	CAREFUL: The transpose is `a.'` while `a'` means complex conjugate.
`xRow*yRow`	Error message, because two row vectors cannot be multiplied.
`xRow*aCol`	Vector product: row times column vector.
`aCol*xRow`	The vector product is usually not commutative.
`aCol(1)`	Obtain a specific element of a vector.
`aCol(1) = 9`	Change a specific elements of a vector.
`xRow(2) = 0`	Set one element of zero. This is not the same as ...
`xRow(2) = []`	deleting one element of the vector. Compare the result to the line above.
`xRow(end)`	Obtain the last element of x.
`xRow(end+1) = 8`	Add an entry at the end of the vector.
`xRow=[xRow 7]`	Grow the vector *xRow*. Can be done several times using `[cursor up]`

Matrices

`zMx=[1 2 3; 4 5 6; 7 8 9]` creates a 3×3 matrix

`zMx(1,2)`	Before hitting return, predict the result. Remember: $\boxed{\text{row, column}}$
`zMx*aCol`	Matrix times vector.
`zMx*xRow`	If the dimensions do not match you get an error message.
`eye(3)`	Identity matrix: Multiply any vector or matrix with it ...
`eye(3)*aCol`	... and it remains the same. Try this for `xRow, zMx`.

Semicolon and colon

`a+b;`	ASuppress output. This seems to make no sense here, but ...
`d=a+b;`	... virtually all lines in MATLAB end with a semicolon.
`1:5`	The colon operator produces a vector in steps of 1.
`1.2:5.5`	Remaining fractions in the interval are ignored.
`xRow(1:2)`	Main use of the colon operator: to access a range of elements.
`xRow([1 2])`	These two things are the same.
`xRow([3 1 2])`	We can even change the order of elements.
`xRow(1:end-1)`	Get all but the last elements – an often used application.

More on vectors and matrices

`uMx=[aCol bCol]`	Create a matrix by concatenating column vectors.
`vMx=[zMx aCol]`	We can even concatenate matrices and vectors.
`load stocks.mat`	Load some sample data: returns of 10 different stocks.
`size(stocks)`	Obtain the dimensions of our data set: number of rows, columns.
`stocks(1:end,1)`	First column only, i.e. the returns for stock number 1.
`stocks(:,1)`	Shortcut for first column.
`stocks(2,:)`	Second row only, i.e. the returns of all stocks on day 2.
`stocks(:,[2 4 7])`	Subset of stocks number 2, 4 and 7.
`mean(stocks)`	Average return per stock: make use of matrix-compatible function.
`std(stocks)`	Daily standard deviation per stock.

True or false?

`1<2`	This is of cause true. In MATLAB: `true=1` and ...
`0==1`	`false=0`. Note the double equality sign for comparisons.
`aCol<bCol`	Vectors and matrices are being compared element-by-element.

Plots

`plot(stocks)`	All 10 stock returns in one.
`plot(stocks(:,1))`	Just the first stock.
`plot(stocks(:,1),stocks(:,2),'o')`	Scatter plot. Stocks one and two are correlated.
`plot(stocks.')`	Undesired result: MATLAB expects data series in rows.
`plot(cumsum(stocks))`	Cumulative returns for all stocks.
`hist(stocks(:,1),10)`	The command `hist(x,n)` produces a histogram with n bins.
`hist(stocks)`	Ten histograms in 1 plot.
`diary off`	Turn off the diary. It is in `intro.txt` in the *current directory.*

TIP: Now is a good time to have a look at Appendix D.1, which lists all MATLAB symbols.

2.4. Getting help

- In the command window, use `help <command>`
- For more general help, type `lookfor <concept>`
- Use the menu Help/MATLAB Help
- You can also get help by right-clicking any command and choosing Help on Selection.
- Demo programs: use `demo` to get a list. Interesting demos include `travel`, a travelling salesman problem.
- All of the excellent MATLAB user guides are available on `www.mathworks.com`

2.5. Useful web resources

- Programs
  - `www.mathworks.com`
  - `freemat.sourceforge.net`
    A MATLAB compatible open source program with editor and graphics support.
  - `www.gnu.org/software/octave/`
    Homepage of Octave, the free program that is compatible to MATLAB.
  - `lavica.fesb.hr/octave/octave-on-line_en.php`
    Online version of Octave.
- Code
  - `www.matlabcentral.com`
    MATLAB file exchange, newsgroups and blogs.
  - `www.wilmott.com`, `www.nuclearphynance.com`, `mathfinance.com`, `mathfinance.cn`
    Online quant finance communities. A lot of code is also useful for economists.

  - `www.statsci.org/matlab.html`
    Statistical toolboxes for MATLAB.
  - `www.spatial-econometrics.com`
    A comprehensive and well-documented econometrics toolbox.
  - `www.kevinsheppard.com`
    Homepage of the MFE financial econometrics toolbox.
  - `home.tiscalinet.ch/paulsoderlind`, `www.mathworks.com/moler`
    Homepages of distinguished researchers with useful resources.

- Some further reading:
  - `history.siam.org`
    The history of numerical analysis and scientific computing.
- Data → page 48.

2.6. Exercises

Exercise 2.1. Calculate

(a) $1/10^{-6}$ (b) $\sqrt[4]{22}$ (c) 4^{3+2} (d) $\frac{4+7}{12-2}$ (e) $(12-5)^{4/3}$ (f) $\frac{2^2}{3^3}$ (g) $\ln(\frac{2+3}{4-1})$

Exercise 2.2. Working with functions

(a) Calculate $\ln e^x$ and $e^{\ln x}$ for $x = 2; x = 10$ and $x = 20$. Are the results equal to x, as expected? *Hint:* use `format long`

(b) Produce a neat table (i.e. a matrix) with the values of the sin and cos for 0, $\pi/2$, $\ldots 2\pi$. Produce a vector of arguments in the first step, then the matrix with results.

(c) Produce the value of e (the Euler number).

Exercise 2.3. (Use of the colon operator)

(a) Produce a row vector x with the integers from 1 to 10.

(b) Produce a 5×5 matrix with the integers from 1 to 25. *Hint:* combine the matrix from five row vectors.

(c) Produce a multiplication table by calculating the outer product of x with itself (i.e. multiply x with x in such a way, that you get a matrix as result).

Exercise 2.4. Start with `load('count.dat')`. This data contains hourly counts of number of vehicles that pass at three different observation points.

- (*a*) Produce a matrix c1, which is the top left 12×2 submatrix of `count`.
- (*b*) Produce a vector c2, which is the 2nd row vector of `count`.
- (*c*) Produce a vector c3, which is the 3rd column vector of `count`.
- (*d*) Produce a matrix c4, which consists of the 1st and 3rd columns of `count`.
- (*e*) Multiply `count` with a suitable vector such that the result contains the sum of the number of cars over the whole day for each of the three observation points.
- (*f*) Produce a matrix c5, which contains the data for all three observation points for the even hours of the day (i.e. 2 a.m., 4 a.m. until midnight).
- (*g*)$*$ It turns out that the counts for observation point 1 and 2 have been swapped for the evening hours, starting at 18:00 hrs. Create a new matrix c6 that corrects this mistake.

Exercise 2.5. Start with `load stocks.mat`, see the interactive intro to MATLAB.

(*a*) Calculate the correlation matrix of the first, the third and the seventh stock.

(*b*) Create a matrix s1, in which you eliminate day number 27 from the sample.

(*c*) Create a vector s2 that contains the weekly returns from the first week (5 trading days).

(*d*) Create a matrix s3 that contains the five weekly returns from the first five trading weeks by concatenating five vectors of weekly returns.

Exercise 2.6. Produce a vector `x` that contains 100 standard-normal random numbers.

(*a*) Calculate the mean of the first twenty elements, then for the second twenty elements and so on.

(*b*) Repeat (a) for the variance. Use the help function to find the necessary command.

(*c*) Create two vectors `y,z` that contains the first/last 50 observations. Calculate the covariance of `y` with `y`, `y` with `z` and `z` with `z`.

(*d*)$*$ Repeat (a) to (c), but do not use MATLAB's built-in commands for the mean, the variance and the covariance. Use vector multiplication instead.

Exercise 2.7. Start with `A=magic(5)`. This creates a 5×5 magic square.

(*a*) Produce a matrix B, which is the top left 3×3 submatrix.

(*b*) Produce a 5×2 matrix C, which consists of the 1st and 4th column vectors of A.

(*c*) Explain briefly (in a comment) what the command magic does. (Use `help`).

(*d*)* Produce a vector w that contains the diagonal elements (a_{ii}) of A.

3. Basic elements of the MATLAB language

References: Hanselman and Littlefield; Mathews and Fink, section 6; Brandimarte appendix A; Getting started with MATLAB, section 2; Opfer

3.1. Variables

3.1.1. Facts about variables

Variables are the building block of every computing language. They work much like in "paper mathematics". To square a specific number, e.g. three, we write 3^2. To express that any given number x be squared, we write x^2.

In MATLAB, every variable has a value. Think more of a "variable" or a "placeholder" than of an "unknown". Variables live until cleared by the command `clear` or MATLAB is shut down.

Implicit declaration When you enter `a=1`, two things happen: First, a variable called `a` is created and second, the value of 1 is *assigned* to `a`. Some programming languages require that variables be defined ("declared") before use; this is not the case in MATLAB.

TIP: Some useful commands: `who, whos, clear, disp`

Naming The big difference to "paper mathematics" is that the name of a variable can be as long as 63 characters in MATLAB. All letters of the alphabet and numbers are allowed, however no spaces, umlauts or accented characters. Some names should not be used: `i` and `j` (reserved for complex numbers) and the names of MATLAB commands. Long names make variables self-explaining (e.g. `output` instead of `Y`).

Names in MATLAB are *case sensitive*, so `Output` would not be the same as `output`.

TIP: It is a good idea to adopt a personal *naming convention* right from the start. Two popular conventions are "lowercase underscore" (e.g. `is_good_student`) and "capitalized words" (e.g. `IsGoodStudent`). See section 12.2.2.

Decimal	0	1	2	3	4	5	6	7
Binary	000	001	010	011	100	101	110	111

Table 3.1.: All 3-bit binary numbers.

bit	component	range
1	sign	+/-
2-12	exponent w/ sign	$[2^{-1022}\ 2^{1024}]$
13-64	significand (mantissa)	$[2^{-52}\ 0]$

Table 3.2.: The IEEE Standard 754-1985 for double precision floating point numbers.

3.1.2. Floating point representation and numerical precision

Computers use bits to store and manipulate information. A bit has only two states: on or off. They correspond to the digits zero and one. Using several (e.g. 16) bits, one can store information in the form of binary numbers. *Example:* With three bits, one can store the numbers from 0 to 7, see Tab. 3.1.

The binary system is only defined for positive integers. To be able to use real numbers, possibly very large ones, computers use the *floating point* representation:

$$x = \pm m \times 10^{\pm k} \tag{3.1}$$

$$\mathtt{x} = \pm\,\mathtt{m}\ \mathtt{e}\pm\mathtt{k} \tag{3.2}$$

where m is called significand (or mantissa) and k is called exponent. The second line shows how MATLAB outputs floating point numbers, e.g. `1.23e-2` for 1.23×10^{-2}.

Example: floating point representation

`1.23e-2`	Enter a small number in floating point notation.
`7.56e4`	A large number.
`format shorte`	Output in floating point notation.
`1.1+2.2`	Note: `e00` means $\times 10^0$, i.e. "times 1".

Today's standard computer arithmetics, which is also used by MATLAB, employs 64 bits to store one floating-point number (see Tab. 3.2). It is defined in the IEEE Standard 754-1985[1]. The maximal and minimal numbers that can be expressed *precisely* in this standard are determined by the significand. They are $2^{52} = 4.5 \times 10^{15}$ and $2^{-52} = 2.2 \times 10^{-16}$. The latter number is sometimes called *eps* from the Greek letter epsilon (ϵ) that usually denotes infinitesimal quantities. MATLAB provides a function called `eps` that returns `2.2204e-16`.

[1]A simplified version of the floating point standard is presented at `tinyurl.com/IEEE754`.

The maximal and minimal exponents are: 2^{-1022} and 2^{1024}, i.e. 2.225×10^{-308} and 1.798×10^{308}. They can be reproduced using the functions `realmin` and `realmax`. The smallest number that MATLAB can handle at all is `eps*realmin=4.94E-324`. Any number smaller than that is seen as zero.

Example: limited precision of floating point numbers

`format long`	Display all 15 digits.
`4.9+3.3`	A rather simple expression. Guess the result before hitting return.
`a=1+1E-16;`	Add two numbers; one comparatively smaller than the other one.
`a-1`	We would expect `1E-16`, however, MATLAB returns 0.

NOTE: For more on precision, see sections 9.2 and 13.

3.1.3. Working with variables

Types. The *type* of a variable describes (a) the type of information that it stores, e.g. numbers or text and (b) the dimension, e.g. scalar, vector or matrix. In MATLAB, every variable has a type, which is assigned automatically at creation. Some operations are only possible or sensible for certain types. As a type cannot be seen from a variable's name, it is your task to ensure the correct type of each variable. For this reason it is also a bad idea to change the type of a variable during the course of a program (even though it is possible).

Example: Types and type mismatches

`a=[1 2]`	Create a vector with 2 elements
`b=[1 2 3]`	Create a vector with 3 elements
`a+b`	MATLAB error: `Matrix dimensions must agree.`
`t1='one'`	Create a text variable
`t1+b`	Strange results when mixing types
`'one' + 'two'`	Even stranger results
`whos`	Produce a list of variables with type and dimension

Matrices

Matrices are the native variable type of MATLAB (in fact, the name stands for "matrix laboratory"). MATLAB uses the standard notation for matrices, in which the dimension is quoted as *rows*×*columns*. Similarly, an element of a matrix is accessed as (*RowNumber, ColumnNumber*).

Matrix dimensions and element position: *row* × *column*

1,1	1,2	1,3
2,1	2,2	2,3
3,1	3,2	3,3

(*a*) `(RowIndex, ColumIndex)`

1	4	7
2	5	8
3	6	9

(*b*) `(SingleIndex);`

Figure 3.1.: Two alternative ways to access the elements of a matrix: by row index and column index or by single index.

To create matrices, use the following commands:

- Create a matrix from data: `Zmatrix = [1 2 3; 4 5 6; 7 8 9];`
- Create from column vectors: `Zmatrix = [aCol bCol cCol];`
- Create from row vectors: `Zmatrix = [xRow; yRow; zRow];`
- Create from same vector: `Zmatrix = repmat(vector, nRows, mCols)`
- Standard matrices: `eye(n)` (identity), `zeros(m,n), ones(m,n)`
- Matrices of random numbers: `rand(m,n), randn(m,n), randi(imax, m,n)`

To manipulate matrices, use the following commands (see also Fig. 3.1):

- Access an entry by row and column: `Z(RowIndex, ColumnIndex)`
- Access a whole row: `Z(RowIndex,:)`
- Access a whole column: `Z(:,ColumnIndex)`
- Delete a row: `Z(RowIndex,:)=[]`
- Grow a matrix by adding a column vector: `Z=[Z aCol]`
- Get the dimensions of a matrix: `size(Z)`
- *Access an entry by single index (rarely used): `Z(SingleIndex)`
  The commands `ind2sub()` and `sub2ind()` convert to and from single notation.

TIP: Useful commands: `sum(X)`, `min(X)`, `max(X)`, `mean(X)`, `sort(X)`

Example: Working with matrices

`load stocks`	A sample data set of simulated stock returns
`stocks(:,1)`	Returns of first stock
`myPF=[1 4 7]`	My portfolio contains stocks number 1,4 and 7
`stocks(:,myPF)`	Returns of stocks in my portfolio
`myWt=[0.2 0.7 0.1].'`	My portfolio weights
`stocks(:,myPF)*myWt`	Returns of my portfolio

Example: System of linear equations $Ax = b$

`A=[-1 3; -1 1];`	Parameter matrix
`b=[-6 2].';`	Result vector
`rank(A)==min(size(A))`	Does the matrix A have full rank?
`x=inv(A)*b`	Solution.

Example: Covariance of two random numbers

`N=1000;`	Good programming style: define parameter first
`X = randn(N,2);`	Produce matrix containing two random numbers
`X.'*X/(N-1)`	Calculate the covariance matrix (supposing zero mean)
`cov(X)`	Zero mean assumption was wrong.
`Y=X-repmat(mean(X),N,1)`	Subtract mean from each row.
`Y.'*Y/(N-1);`	Correct covariance matrix

Vectors

MATLAB has in fact no special provisions for vectors. A row vector is seen as a $1 \times N$ matrix, and a column vector as a $M \times 1$ matrix. This matrix view bears the advantage that we can use all the matrix commands for vectors, as well.

Commands for creating vectors:

- Create row vector: `xRow=[1, 2, 3]` *or* `xRow=[1 2 3]`
- Create column vector: `aCol=[5; 6; 7]` *or* `aCol=[5 6 7].'`
  NOTE: The latter version is widely used.
- Create row vector with with M elements: `linspace(start, end, M)`

Commands for manipulating vectors:

- Concatenate row vectors with space: `[xRow yRow]`
- Concatenate column vectors with semicolon: `[aCol; bCol]`
- Access/delete the n-th element: `x(n)` and `x(n)=[]`
- Access last element: `x(end)`
- Get the length of a vector: `length(x)`
- Grow an existing column vector: `aCol=[aCol; 7]` *or* `aCol(end+1)=7`
- Convert any matrix or vector into a column vector: `y=x(:)`
- Vector plus scalar: in standard mathematics, the operation $\mathbf{x} + \alpha$ is not defined. In MATLAB, the value of α is added to every element of the vector $\mathbf{x}$.

Vector products are widely used in MATLAB to calculate sums.

Sum of sqaures	$x'x = \sum_{i=1}^{n} x_i^2$	`SSE= u.'*u`
Weighted sums	$x'y = \sum_{i=1}^{n} x_i y_i$	`E = probability.' * outcome`

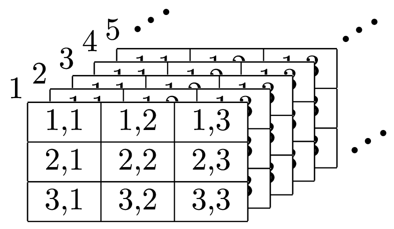


Figure 3.2.: A three-dimensional array can be seen as a time series of matrices.

Example: Reproduce the famous Fibonacci series using a growing vector.

`fib = [1 1]`	Start with the first two numbers as defined
`fib(end+1)=fib(end-1)+fib(end)`	New element is sum of last 2 elements.
`[cursor up] [return]`	Repeat as often as you like.

^{M}**The colon operator** MATLAB has a special operator for creating vectors: the colon operator. There are two syntaxes to use it.

Syntax of the colon operator

`1:5`	*min* : *max* produces a vector in steps of 1.
`1:0.5:5`	*min* : Δ : *max* steps of $\Delta = 0.5$.
`5:-1:0`	*Note:* Δ can also be negative.
`1.2:4.5`	If the range $max - min$ is not integer, the last fraction is discarded.

Datasets

Datasets are convenient way to work with cross-sectional data matrices, where each column corresponds to one variable. A good example is provided by MATLAB's cereal data:

Datasets

`load cereal`	Data set containing cereal nutrition data

```
cereal = dataset(Calories,Protein,Fat,Sodium,...
    Fiber,Carbo,Sugars,'ObsNames',Name)
```

`cereal`	Output the dataset in nice table
`cereal.Fat`	Access one column on the dataset

*Arrays

Example 2. Imagine you are performing a portfolio analysis with historic data. For every quarter in the last year, you have a correlation matrix of all assets, i.e. 4 correlation matrices as depicted in Fig. 3.2. Is there a way to store these in a neat and tidy way? The answer is: use an array. An array is a variable with more than two dimensions. They are used exactly like matrices, just with more indices. In our case, three dimensions do the job: two for the correlation matrix and one for time.

Example: Four quarterly covariance matrices

`retn=randn(252,3);`	Produce some return data (252 trading days) ...
`prices=cumprod(1+retn*0.1);`	... and prices.
`Z1=cov(retn(1:63,:));`	Covariance matrix for first quarter.
`Z2=cov(retn(64:126,:));`	Four variables `Z1` to `Z4` would be cumbersome.
`CX(:,:,1)=Z1; CX(:,:,2)=Z2`	Store everything in an array
`CX(:,:,3)=cov(retn(127:189,:))`	Third quarter
`CX(:,:,4)=cov(retn(190:252,:))`	Fourth quarter
`CX(:,:,1)`	Retrieve first covariance matrix
`CX(1,2,:)`	Time series of covariance of stock 1 with stock 2
`squeeze(CX(1,2,:))`	Create a 4x1 vector

TIP: Many of MATLAB's built-in functions work even with arrays, e.g. `mean()`. In the case of the mean, one can even specify along which dimension it should be taken. Try these three commands to see the difference: `A=randn(3,3,10);  mean(A)  mean(A,3)`

*Structures

Structures provide an elegant way to combine variables of different types. They contain any number of *fields*, each of which can be of different type and dimension. Their main use is to reduce complexity and to make the code more readable; e.g. when calling functions. This is used by many toolboxes. Structures are created implicitly using the dot. To access a filed use *structure.field*.

Example: Combining different types of information

`obs.data=[6.0 5.5 4.0 5.5]`	Numerical data (e.g. student grades)
`obs.info='Student grades'`	Text information about the data

Several structures of the same composition may be combined to a structural array, e.g. if you want to build a company database:

```
comp(1).name='IBM'
comp(1).dividend=0.12
comp(2).name='Apple'
comp(2).dividend=0.0
```

*Cell arrays

Cell arrays are your choice if you want to store information of different structure and dimension. For more on cell arrays, see page 50.

3.2. Arithmetic operators

Arithmetic operators – these are plus, minus, divide and power – behave exactly like in paper mathematics. When we calculate 2+3*5, we know that the result is 17, because the multiplication has to be performed first. This concept is called *operator precedence*. Tab 3.3 shows the precedence rules for all operators. Unary operators are operators that have only one operand. An example is `-2*-3`, which is (correctly) +6 in MATLAB.

TIP: A major sources of errors is the fact that some brackets are not written in paper mathematics. *Examples:* $\frac{a+b}{c+d}$ which is `(a+b)/(c+d)` and 2^{a+b}, which is `2^(a+b)`.

1	()			
2	'	^		
3	+	-	~	(unary)
4	*	/	\	
5	+	-		
6	:			

7	<	<=	>	>=	==	~=
8	&					
9	|					
10	&&					
11	||					

Table 3.3.: Operator precedence in MATLAB for all arithmetic, relational (sec. 4.2) and logical (sec. 4.4) operators.

^{M}Dotted operators and vectorized calculations

MATLAB goes beyond paper mathematics in offering element-by-element operators for matrices. These are denoted with a dot in front. This is relevant for three operators (plus and minus are always element-by-element): `.*` `./` `.^`

Dotted operators make a calculation/function vector compatible.

Examples: Dotted operators

`Y=[1 2; 3 4].';`	Some matrix
`Y*eye(2)`	Standard multiplication
`Y.*eye(2)`	Element-by-element multiplication, also denoted $Y \odot Id$
`Y^2`	Matrix-wise $Y^2 = Y * Y$ (only for square matrices)
`Y.^2`	Element-by-element (y_{ij}^2) (possible for all dimensions)

`x=randn(1000,1);`	Standard-normal distributed vector $x \sim N(0,1)$
`y=x.*x;`	χ^2 distributed vector $y \sim \chi_1^2$

`q=0.5;`	Define some $0 < q < 1$.
`sum(q.^(0:20))`	Verify $\sum_0^\infty q^k = 1/(1-q)$ by summing up the first 21 elements.

Example: Relative quantities

`load count.dat`	Same traffic data as in Section 2.
`count(:,3)./count(:,2)`	Traffic at location 3 relative to traffic a location 2.

3.3. Programs

A program is a sequence of MATLAB commands. It is saved in plain text in a file with extension `.m` (also called *M-File*). MATLAB has a convenient editor, but you could use any word processor to write a program. This section and PC lab 3.5 describe how to edit program code. See chapter 12 on the important topic of planing and writing programs.

Computer programs often start as small projects and evolve into huge, confusing collections of code filling dozens of pages. It has therefore proven useful to stick to a standard structure, like the traditional structure for MATLAB programs:

1. First line: the name of the program
2. Preamble: a short comment explaining the program
   At the end of the preamble add the author's name, e-mail and the data
3. First block: all constants – so they are easy to find and change
4. Load and immediately convert all data.
5. Divide the main program in blocks; start each block with a comment
6. The output should be done at the end (this includes saving the results)

Figure 3.3.: Traditional structure for programs

Comments explain a program to the human reader. MATLAB ignores all text between a percent sign (`%`) and the end of the line. Comments are either place at the end of each line or in a separate line above. See chapter 12 on how to formulate comments.

A double percent sign (`%%`) starts a new MATLAB "cell". These cells are a convenient way to structure a program. Using the button, one can run programs cell-by-cell.

TIP: Practitioner's tips for writing programs

- Save frequently and use versions after a major step, e.g. `MyPrg01.m`, `MyPrg02.m` ...
- A clean code layout helps reading and understanding a program.
- Adhere to the rules of good programming style, see chapter 12

```
% myfirst.m
% This is our first program
% peter.gruber@usi.ch, MATLAB class 2006-2014

% Store the cost of food and beverages in two variables
food=50;
beverages=100;

% Calculate the total cost
```

```
cost=food+beverages;
disp(cost)
```

3.4. Functions

3.4.1. Built-in functions

The built-in functions of MATLAB implement most standard mathematical functions and some basic statistics. Functions can be called in the command window, in programs or in other functions.

TIP: Useful MATLAB functions are: `sqrt(), exp(), log(), log10(), sign(), abs(), round(), ceil(), floor(), sum(), min(), max(), mean(), var()`

TIP: For a list of all MATLAB built-in functions, open Help/MATLAB Help and navigate to MATLAB/Functions – By Category.

3.4.2. ^{M}Inline functions

A quick way to define a function is the so-called *function handle*. This permits defining a function "on the fly" (i.e. within a program) and immediately using it. It is even possible to define and use an inline function in the interactive mode.

Example: inline functions

`sq=@(x)x^2`	Create a function `sq` with one input argument (x)
`sq(3.2)`	Using the function `sq`.
`utilE=@(c,a)1-exp(-a*c)`	Exponential utility: $u(c) = 1 - e^{-\alpha c}$.
`utilE(100,2)`	
`invlog=@(x)1/log(x)`	Inline functions can make use of other functions
`ssq=@(x,y,z)sq(x)+sq(y)+sq(z)`	

***Recasting functions.** In economics, we are often interested in what happens to a function if we change one argument, keeping the others constant (*ceteris paribus*). This can be easily done in MATLAB using inline functions. Take the exponential utility. We want to fix the risk aversion $\alpha = 2$ and make a plot of the utility as a function of the consumption.

Example: casting multiple-argument functions as one-argument functions

`myRand=@(M)randn(10,M)`	Create random matrices with 10 rows and M columns.
`myUtil=@(c)utilE(c,2)`	New function `myUtil()` depends only on c; α is fixed to 2.
`fplot(myUtil,[0,10])`	MATLAB's `fplot()` requires functions of 1 argument.
`fplot(utilE,[0,10])`	This does not work.

Similar considerations apply for MATLAB's numeric integration functions (to integrate along one dimension only) or optimization functions.

3.4.3. User-defined functions

User-defined functions are plain text files and are edited and commented just as MATLAB programs. The file name must be `FunctionName.m`, otherwise MATLAB will not find the function. The first line has to start with the keyword `function`. Again, it has proven useful to stick to a traditional structure when writing a function.

1. First line: `function` *result* = *FunctionName* (*arguments*)
2. Preamble – this is a short comment that explains what the function does. This will be used as help text for the function.
3. Explanation of input and output parameters
4. Usage example.
5. Divide the calculation in blocks; start each block with a comment
6. Assign the function value to the variable *result* from the first line.

Figure 3.4.: Traditional structure for user-defined functions

Example 3. The payoff of one CHF in a bank account using continuously compounding interest rates is a function of the interest rate r and the time t. Formally:

$$R: \quad \mathbb{R}^2 \to \mathbb{R}, \qquad R(r,t) = e^{rt} \tag{3.3}$$

In MATLAB, we can implement this function of two scalars as follows.

Example: continuously compounded interest `ccinterest.m`

`function y = ccinterest(r,t)`	Function definition
`% Continuously comp. return on 1 CHF`	Explanation also used for `help ccinterest`
`% INPUT   r   1x1 .. interest rate`	Explain type and meaning of input ...
`%         t   1x1 .. time in yrs`	parameters (inclding units) and ...
`% OUTPUT  y   1x1 .. payoff`	of the output parameters.
`% USAGE   ccinterest(r,t)`	Usage example.
`% peter.gruber@usi.ch, 2011-09-01`	Author/contact/date
`y = exp(r*t);`	Assign a value to `y`, to define the output
	What happens if we forget the semicolon?
`end`	Optional: mark the end of function

Why use functions? This is a valid question: we could have used a simple program to for this calculation and it would have been much shorter. However, functions provide unique advantages: They are easily re-used and thus avoid copy-paste errors when duplicating code. They can be easily shared among your own projects and with co-authors. They make a program more readable by hiding "boring" code (e.g. data conversions). Finally, they help to structure a project and make testing and finding errors easier.

Tips for writing functions

- Apply the same rules for naming variables to naming functions.
- Functions should not contain any constants to be changed. Everything that can be changed should be an argument.
- Functions should not produce output and should not load data (exceptions apply).
- It is possible to put more than 1 function into an m-File. All but the first one are *private functions* that are not accessible from outside. Useful for distributing code.

^{M}Multiple outputs

User-defined functions can have any number (including zero) of input and output arguments. For multiple output arguments, there are two solutions.

```
function nb = neighbours1(x)
% Neighbours of a number.
% Version with output in structure
% INPUT    x  1x1 ... a scalar
% OUTPUT  nb struct
%          nb.p ... predecessor
%          nb.s ... successor
nb.p=x-1;
nb.s=x+1;
```

Version 1 with output in a structure

```
function [p, s] = neighbours2(x)
% Neighbours of a number.
% Version with 2 output variables
% INPUT    x 1x1 ... a scalar
% OUTPUT   p 1x1 ... predecessor
%          s 1x1 ... successor

p=x-1;
s=x+1;
```

Version 2 with two output variables

To call the first function, use `myNb = neighbours1(2)` and then `myNb.p` and `myNb.s`, to call the second version use `[first, second] = neighbours2(2)`.

Discussion. At first sight, the two functions look the same. There is, however, a big difference in how they are used. In the second case, we need to know the order of the output arguments. Whatever variable is first will be the predecessor, the second one the successor. For a long list of outputs, it may be cumbersome to remember the ordering. In the first case, we have to use the structure, which may involve a bit more coding. However, the structure is easier to understand and helps avoid errors.

TIP: Use the terms *input argument* and *output argument* to avoid confusion.

^{M}Vectorized functions

Nearly all built-in functions can use matrix input and are executed element-by-element. User-defined functions have to be made specifically vector-compatible. Quite a few MATLAB commands require vector compatible functions, e.g. `fplot` (plotting), `quad` (numeric integration) or `lsqnonlin` (optimization). Usually, we use dotted operators (sec. 3.2) to make a function vector compatible, but in a few difficult cases, we have to rewrite the function, e.g. by using for-loops.

Example: matrix/vector-compatible functions

`a=[1 3 5]`	An input vector
`log(a)`	The logarithm is applied to each element of a
`sq(a)`	Applying our function to a vector produces an error message.
`sq2=@(x)x.^2`	Just add one dot to vectorize it.
`sq2(a)`	Works.

3.5. PC-Lab: The first program and the first function

Create a program from scratch in the program editor

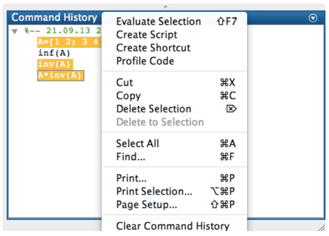
1. Type `edit newprogram.m` in the command window. A message informs you that this file does not exist and asks you whether you want to create this file. If you accept, MATLAB creates a new program file in the current directory.
2. Now you can type a program. Note the automatic syntax coloring and indenting.
3. To run the program, use this button from the toolstrip.
4. Alternatively, enter the program's name (`newprogram.m`) in the command window. In this case, you need to save the program first.

Cell mode

1. Load `celldemo.m`. The first cell is always shaded yellow. This is the current cell.
2. Click into the second cell. Now the second cell is the current one and yellow.
3. To execute a program cell-by-cell, click again into the first cell and use the "Run and Advance" button ( ) in the edit toolstrip.
4. Add a new cell at the end of the program. Press return and type `%% New cell`

The first function

1. Type `edit neighbours.m` in the command line and accept to create a new file.
2. Copy the first version of the function from page 31. Save and close it.
3. Test the function from the command line as indicated on page 31.



```
%-- 21.09.13 2
A=[1 2; 3 4
inf(A)
inv(A)
A*inv(A)
```

Figure 3.5.: Create a MATLAB script from the command history.

Create a program from the command history

In an evolutionary approach, you may want to start by trying a few commands in the command window. Once satisfied, create a program directly from the Command History.

1. Select the desired commands using control-click.
2. Right-click as shown in Fig. 3.5 and select Create Script.
3. The result is a new program. Add comments and don't forget to save.

Convert ccinterest.m to a vectorized function

It may be interesting to calculate the continuously compounded interest for several durations or several interest rates in one go, e.g. in order to produce a plot. Maybe this is already possible ...

`ccinterest(0.05, 1)`	One interest rate, one duration.
`ccinterest(0.01:0.01:0.1, 1)`	Several interest rates, one duration.
`ccinterest(0.05, 1:15)`	One rate, durations from one to 15 years.
`ccinterest(0.01:0.01:0.1, 1:15)`	This, however, does not work. Why?

First, we have to think about the desired functionality of our vectorized function. Which outputs should be created if (a) the rates, (b) the durations or (c) both are vectors? One possible solution is to interpret (c) in such a way that the first rate should be combined with the first duration, the second with the second and so on ... In this case, the solution is to use the dotted multiplication operator, simply changing one line in the function.

`y=exp(r.*t);`	Element-by-element multiplication.

If you make such a change, also adapt the documentation:

```
% INPUT   r   1x1 or 1xN .. interest rate
%         t   1x1 or 1xN .. time in yrs
%         if r and t are vectors, the first r is evaluated at the first t and so on.
```

Now we can test our function

`ccinterest(0.01:0.01:0.1, 1:15)`	Still not working.
`ccinterest(0.01:0.01:0.1, 1:10)`	This works, but does it make sense?

3.6. Exercises

Exercise 3.1 (Precision)**.** For small values of d, Matlab gives zero as a result for the following two subsequent commands: `a=100+d; a-100`. Find (by try and error) the largest value of d for which this artifact occurs.

Exercise 3.2. * Some matrix operations have only limited precision. To see this, let
$$A = \begin{bmatrix} 1 & 2 & 3 \\ 4 & 5 & 6 \\ 7 & 9 & 8 \end{bmatrix} \text{ and } B = \begin{bmatrix} 2 & 2 & 4 \\ 1 & 3 & 5 \\ 2 & 4 & 8 \end{bmatrix}.$$

(a) Verify the identities for the transposed on page 172 and the trace on page 172.
(a) Verify the identities for the inverse on page 171 and the determinant on page 173.
Hint: Use `format long` to see possible tiny differences or check for equality using the comparison operator `==`.

Exercise 3.3 (Inline functions.)**.** Write a short program that implements the following utility functions as inline functions: Log utility: $u(c) = \ln(c)$, power utility: $u(c, \gamma) = \frac{c^{1-\gamma}-1}{1-\gamma}$, exponential utility: $u(c, \alpha) = 1 - e^{-\alpha c}$, quadratic utility: $u(c, a) = c^2 - ac$
(a) Use the command `fplot()` to plot the utility functions for consumption levels between 0.5 and 10 for sensible values of the parameters.
(b) Make another plot with the power utility and values of γ of 1.1, 2, 5 and 10 as well as the log utility. Use `hold on` to produce one plot with all five curves.
(c)* A two period utility function is defined as $U(c_1, c_2, \delta, \cdot) = u(c_1, \cdot) + e^{-\delta} u(c_2, \cdot)$, where $u(c, \cdot)$ can be any of the above utility functions. Find a concise way to write a two-period utility function using the power utility. How many arguments will this function take?

Exercise 3.4. Write functions that perform the following tasks using the standard structure of a function as described in in Fig. 3.4.
(a) Sum of all integers up to N. Do not use the Gauss sum formula $\frac{N(N-1)}{2}$.
(b) Calculate $\sum_{k=1}^{N} k^2$.
(c) Calculate $\sum_{k=1}^{N} k^\alpha$ using two input arguments.

Exercise 3.5 (Black-Scholes formula)**.** The famous Black-Scholes formula for option pricing takes five parameters: today's stock price S, the strike price K, the duration t, the interest rate r and the volatility σ. The price of a call option is then:

$$C = SN(d_1) - Ke^{-rt}N(d_2) \tag{3.4}$$

with $d_1 = \frac{\ln(S/K)+(r+\frac{1}{2}\sigma^2)t}{\sigma\sqrt{t}}$ and $d_2 = d_1 - \sigma\sqrt{t}$. The symbol $N(\cdot)$ denotes the c.d.f. of the standard normal distribution, with $N(x) = 0.5 + 0.5\,\mathrm{erf}(x/\sqrt{2})$. The error function `erf()` is available in MATLAB.

(a) Write a function `StdnCdf`, which calculates $N()$ using `erf()`. Follow the standard structure for user-defined functions in Fig. 3.4.

(b) Write a function `BlackScholesCall` that makes use of `StdnCdf` and calculates the price of a call option, given S, K, t, r and σ. Choose sensible names for the variables. Test your function with sensible values.

$*$(c) Would it make sense to make `BlackScholesCall` vector compatible? With respect to which input parameter(s)? Try to make your solution vector-compatible.

Hand in your solution as **one file** using private functions if necessary.

Exercise 3.6 (Up to you.)**.** Find at least 5 job advertisements in your field in which MATLAB is a requirement.

4. More elements of the MATLAB language

References: Hanselman and Littlefield; Getting started with MATLAB, section 4

4.1. Comparing, branching and looping

Example 4. Toss a coin: *If* you win, you get one Swiss Franc, *else* nothing. Our game has two states: `0=loose 1=win`. How can we flip a coin in MATLAB? There is no command for coin flipping in MATLAB, but we can use `rand`, which produces uniformly distributed random numbers between 0 and 1, and apply a transformation: `round(rand)`.

`round(rand)`	Tosses of a coin: zero or one
`round(rand(1,5))`	Five tosses of a coin: zeros or ones

We want to display "win" *if* the state is win and "loose" *else.* This is our first flow control statement.

Example: If – else – end

`state=round(rand)`	Generate state (1=win, 0=loose)
`if state == 1`	The logical condition is (state == 1)
`  disp('win')`	Executed if the condition is true
`else`	
`  disp('loose')`	Executed if the condition is false
`end`	Don't forget the `end`

Two things happen in the line with the `if` statement. First, a comparison is carried out: Is the variable `state` equal to 1? Second, depending on the result of this comparison, the program continues with different pieces of code. These two concepts will be discussed separately in the following two sections. First we discuss comparisons (relational operators), thereafter flow control, i.e. what code to execute after a comparison.

4.2. Relational operators

Relational operators are listed in Tab. 4.1. Why are they called operators? Because they link two variables (*operands*) and give a new result; just like arithmetic operators.

The image set of relational operators is quite small. It contains only `true` and `false`. MATLAB assigns numerical values to these logical expressions:

`==`	equal	`~=`	unequal
`<`	smaller than	`<=`	smaller or equal
`>`	larger than	`>=`	larger or equal

Table 4.1.: Relational operators in MATLAB. Note the double equal sign (`==`) for equality.

$$\boxed{1 = \text{true} \qquad 0 = \text{false}}$$

Relational operators are matrix compatible in MATLAB. If two matrices (of equal size) are compared, MATLAB produces a matrix of zeros and ones that contain the result of the element-by-element comparison.

`1 + 1` → `2`	An arithmetic operator takes two operands to produce a result
`1 < 2` → `true`	A logical operator does exactly the same.
`[1 3] < [2 2]`	The result is `[1 0]`, because $(1 < 2)$ is true and $(3 < 2)$ false.

TIP: Matrix-compatible *logical functions* in MATLAB: `all()`, `any()`, `xor()`, `isempty()`, `issorted()`, `ismember()`, `isequal()`,

Equality and floating-point numbers.

Floating-point numbers are rarely exactly (up to 10^{-16}) equal. Instead, use a relative difference measure.

`a=11.1+12.2; b=23.3; a==b`	Not true because of rounding errors.
`abs(a-b)/(a+b) < 1E-6`	This relative condition is true.

*Calculations with relational operators

The fact that "false" and "true" have numerical values can be readily used for calculations: to count the number of observations for which a condition is true. Or consider the following transformation: $(x)^+ := \begin{cases} x & \text{if } x > 0 \\ 0 & \text{if } x \leq 0 \end{cases}$, which is used in the payoff of a call option $(S_T - K)^+$.

`data=randn(1,7);`	Produce some data.
`data>0`	Vector of zeros and ones for negative and positive.
`dataIsPos = data>0`	Save result of comparison in new variable.
`sum(dataIsPos)`	Number of positive entries.
`mean(dataIsPos)`	Fraction of positive entries.
`data .* dataIsPos`	Transformation (`data`)$^+$.

*Logical indexing

To extract all positive elements from the vector *data*, use *logical indexing*:

`data(dataIsPos)`	Extract positive elements of `data`. Result is a shorter vector.

4.3. Flow control

4.3.1. Single choice: if – else

The if – else clause makes it possible to selectively execute parts of the code. The full syntax is (parts in square brackets are optional):

```
if condition
        commands ... ;

[elseif condition
        commands ... ;]

[else
        commands ... ;]
end
```

Indenting code Note that the commands between `if, else` and `end` are indented. This is automatically done by the MATLAB editor for all commands that have an `end`, i.e. these commands: `if, case, for, while`. If you have written your code outside MATLAB, you can use `ctr-I` to automatically indent your code. Indenting is especially useful if several of these commands are nested: for every level of indenting, there must be an `end` somewhere. As long as the text is indented, we are still inside one of these clauses. Indenting also makes the code more readable.

Matrix-valued comparisons revisited

Matrix comparisons produce matrix results. Is the if condition met, when the comparison matrix contains some zeros and some ones? Try (`help if`) for the answer:

> "The statements [after the if] are executed if the real part of the expression has *all* non-zero elements."

To check for equality, consider using `isequal()`, which produces a scalar result. Note that `isequal()` also permits comparing vectors of different lengths (which always results in *false*). Directly comparing vectors may lead into the following trap:

Example: some vector comparisons are neither true nor false.

```
a=[1 2 3]; b=[1 2 4]                    Define two vectors
if a==b;disp('equal');end               We expect no output here: a ≠ b.
if a~=b;disp('unequal');end             We would have expected an output here!
a==b                                    Why? Both expressions do not meet the
a~=b                                    requirement of "all non-zero elements"!
isequal(a,b)                            Scalar result
if ~isequal(a,b);disp('uneq.');end      This works!
```

4.3.2. Discrete multiple choice: switch – case

Multiple alternatives could be modeled using cascades of if – elseif – end, but this makes the code difficult to read and evokes repeated evaluations of the condition, which may lead to unwanted results. An alternative is the switch statement. The expression *value* can be a variable, a function or even a short calculation; $c1$ and $c2$ can be numbers or vectors.

```
switch value
    case c1
        commands ... ;
      [case c2
        commands ... ;]
    [otherwise
        commands ... ;]
end
```

Example 2: test for multiple possibilities

```
switch weekday(date)                     Check for multiple discrete possibilities
    case {2,3,4,5}                       MATLAB's week starts on Sunday,
        disp('Grum working days');       thus Monday is day number 2.
    case 6
        disp('Home at 12:00');
    case {7,1}
        disp('Weekend!');
    otherwise
        disp('Your calendar is broken');
end
```

4.3.3. Conditional loops

Consider the following statement: "*While* there is wine in the bottle, I go on drinking." A computer would do the following: drink a sip, check if there is still wine, drink a sip and so on until the bottle is empty. In MATLAB, this is done using the `while` command.

```
while condition
    commands ... ;
end
```

If the *condition* is false in the first place, the loop is never executed. If the condition is always true (e.g. trivially `while 1==1`), the loop is executed an infinite number of times. Thus, the number of executions of a while loop is between 0 and ∞. The main application are so-called iterative algorithms, that find an ever-better approximate solution for a numerical problem ("while the solution is not good enough, continue improving"), see section 8.

TIP: It's a good idea to count the number of executions of a while loop and to provide an emergency exit after, say, 1000 executions. The user can stop an infinite loop by pressing `ctr-C`. MATLAB may take a few seconds to abort the loop.

> `Octave note:` In Octave, there is an alternative loop command using `repeat` ... `until`. This is highly useful and very intuitive. Before using it, consider that it is not compatible with MATLAB.

The classical structure of a while loop

Example 5. We illustrate the classical structure of a while loop with a game of three strikes. Consider the following game: you throw three coins simultaneously. You win, if all three coins show heads. How many tries do you need to win? How does the distribution look like?

1. Initialize counter.	`tries = 0;`
2. Initialize test variable *outside* the while loop.	`win = 0;`
3. Perform comparison and check for the emergency exit.	`while sum(win)<3 && tries<1000`
4. Calculate test variable.	`   win = round(rand(1,3));`
5. Increase counter.	`   tries = tries+1;`
6. Don't forget the end	`end`
	`disp(tries);`

Figure 4.1.: Traditional structure for while loops, see `coins.m`

4.4. Two conditions: Logical operators

Consider the statement: "If you get rich *and* famous, I will marry you, else forget about it." Both conditions for marriage have to be met. The logical operators `and, or` combine two conditions (logical values). They operate on a small set: $\{0,1\} \times \{0,1\} \rightarrow \{0,1\}$. The `not` operator switches between true and false. See Tab. 4.2.

`&`	and	true, if *all* conditions are fulfilled
`|`	or	true, if *at least one* condition is fulfilled
`~`	not	true $\leftrightarrow$ false

Table 4.2.: Logical operators in MATLAB.

Example: marriage conditions

`rich=round(rand)`	Life is ...
`famous=round(rand)`	... random
`if rich & famous`	Check both conditions
`  disp('I will marry you.')`	Both conditions are met
`else`	
`  disp('Forget it.')`	One or both conditions not met
`end`	Both conditions are met

^{M}Short-circuited logical operators

Consider the example `1 | <something>`. This will always be true. However, it may be quite costly to calculate `<something>`. For this case, MATLAB has a second version of `and` and `or`, which have double symbols: `&&` and `||`. They are called *short-circuited*, because the second expression is only evaluated if the result is not known after the first one. They are always faster, but they only work on scalar operands.

Example: short-circuited operators

`1 | sum(eig(rand(1000)))`	This takes quite a while to execute.
`1 || sum(eig(rand(1000)))`	This is much faster.
`0 & sum(eig(rand(1000)))`	This again is slow.
`0 && sum(eig(rand(1000)))`	Much faster: we already know it will be false.
`1 && sum(eig(rand(1000)))`	No improvement here: This can be true or false.
`[0 1] && [0 0]`	Produces an error message as operands are not scalar.

4.5. For loops

Consider the case that you forgot to hand in your problem sets in time. You get the special exercise (we don't punish students) to write 100 times "I will not forget to do the problem set." This is a stupid, repetitive task. Can the computer do it for us? Yes – computers are very good at stupid, repetitive tasks.

Example: special exercise

```
for k=1:100
  disp('I will not forget to do the problem sets');
end
```

In a for-loop, the number of executions is exactly known in advance. In MATLAB, there are a three ways to define for loops, which are all rooted in how to define a vector. The basic syntax of the for loop is:

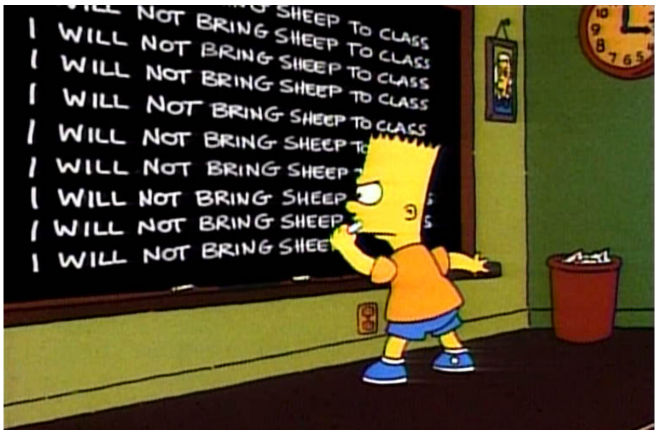


Figure 4.2.: You write 100 times “I will not forget to do the problem set”.

```
for variable =  vector
    commands ... ;
end
```

A for loop executes exactly as many times as there are elements in *vector*. At every execution, the *loop variable* is assigned one value of *vector*, starting with the first one. Most for loops look slightly different. Remember the standard ways to create a vector;

Recap: different ways to create a vector

`[3 5 2.2 1.7]`	Element by element
`1:10`	Increments of 1
`1:T`	Increments of 1 up to `T` (must be defined previously)
`1:0.5:10`	Arbitrary increments
`linspace(1,10,20)`	Fixed number (here 20) of elements

Example: A countdown in a for loop

```
for k=10:-1:1          Count from ten to one
  disp(k);
end
```

Defining a loop with a vector can be useful if you want to loop over a subset of the data for which there is no simple rule (e.g. every second Wednesday of each month). Note that the vector needs to be a row vector.

4.5.1. A few more things about loops

- Do not use `i` or `j` as loop variables. They are reserved for complex numbers.
- Loops can be executed in the command window. MATLAB waits until you enter `end`.

- Loops are comparatively slow in MATLAB (see section 13.2.1), therefore be careful what belongs inside the loop (e.g. execute data conversion before the loop, calculate statistics after the loop).
- There are two special commands with for loops: `break` stops the execution of the loop and leaves it, `continue` stops the current iteration and starts next iteration. Both commands make understanding a program more difficult and can be easily replaced with `if` or `while` clauses. Therefore use of these commands is discouraged.
- MATLAB allows you to change the loop variable within the loop. This usually creates a lot of confusion and should not be done.

4.6. PC lab: Applications of for loops

`diary('forloops.txt')`	Don't forget to start a diary.

Fixed number of repetitions: Running sum of cash flows (present value)

A simple method for valuing an investment is the present value: the sum of all discounted payments: $PV = \sum \frac{CF_i}{(1+r)^i}$. Assume a shopping center that is rented at a fixed rent of y over 20 years. The interest rate be r. For-loops are perfect for implementing a sum. Note that we need to initialize the running sum variable outside the loop.

`r    = 0.05;`	Set interest rate: $0.05 = 5\%$.
`y    = 10000;`	Set rent
`Tmax = 20;`	Contract runs 20 years.
`PV=0;`	*Initialize running sum outside the loop.*
`for k = 1:Tmax`	We receive payments for $Tmax$ years
`  PV = PV + y/(1+r)^k;`	Running sum; every year we add to PV
`end`	Note that when entering a loop in the command window, it will start once you enter `end` and press return.
`disp(PV);`	

Fixed number of repetitions: The distribution of the game of three strikes

A small simulation: We want to produce a histogram of the number of tries until we get three strikes. To do so, we first convert the program `coins.m` into a function called `coin_fn.m`. We then play 1000 games using a for loop and produce a plot.

```
N=1000;                          Define the number of iterations outside
for k=1:N                        We run the game N times
    results(k)=coin_fn;          Result of iteration k is stored in k-th element of results
end
[mean(results) var(results)]     Some nice summary statistics
hist(results);                   And finally a histogram to see the distribution
```

Data-driven number of repetitions: Looping over data

Many empirical applications (see chapter 5) require you to inspect every data point. In such a case, the number of executions of the for loop depends on the size of the data matrix and should therefore be defined as such:

```
data=randn(100+randi(100),1)     Produce a data set of random length (100 .. 200 entries)
for d=1:size(data,1)             Number of loop executions = number of rows in data
  if data(d)<0                   Simple example of data manipulation
     data(d)=0;                  ... negative entries are set to zero
  end
end
```

*Nested for-loops

Sometimes we have to put a for-loop within a for-loop, for example if we want to analyze different versions of a calculation that contains a for loop. We revisit the present value calculation, but now with multiple different interest rates.

```
y    = 10000;                  Set rent
Tmax = 20;                     Contract runs 20 years.
rates= 0:0.01:0.2;             Possible interest rates.
for s = 1:length(rates)        Loop over possible interest rates
  r     = rates(s);                r varies between zero and twenty percent
  PV(s) = 0;                     Initialize the s-th sum.
  for k = 1: Tmax                  We receive payments for the next Tmax years
    PV(s)=PV(s)+y/(1+r)^k;           Running sum; add contribution of year_k to PV_s
  end                            End the loop over the years.
end                            End the loop over the interest rates.
disp(PV);                      Output is now a vector
plot(rates,PV);
```

NOTE: Why do we need the variable s? At first sight, it seems we could have simplified things by writing the first loop as `for r=0.01:0.01:0.2`. This is correct as long as we do not want to save the results. As we need the results for later output, we want to store them in the vector PV. The indices of this vector must to be integer. Therefore we need the loop variable s that runs from 1 to `length(rates)`.

`diary off` Now it is time to mail the diary to yourself.

4.7. Exercises

Exercise 4.1. Write functions that fulfill the following tasks using the standard structure in Fig. 3.4. (*a*) Return the smaller of two input arguments. Do not use `min()`. (*b*) Return 1 if all elements of an input vector are even and 0 else.

Exercise 4.2 (Fibonacci series.)**.** The Fibonacci series starts with $\{1, 1\}$ and all subsequent elements are the sum of the two previous ones. The first few elements are $\{1, 1, 2, 3, 5, 8, \dots\}$. Write a small program or function that ...

(*a*) ... computes the Fibonacci series up to the n-th element.

(*b*) ... computes all elements of the Fibonacci that are smaller than m.

(*c*) ... verifies for the first l elements that the quotient of two neighboring Fibonacci numbers approaches the golden ratio and produces a plot illustrating this fact.

Exercise 4.3. Produce N random numbers and calculate the average. Repeat this M times so that you get M averages. Plot a histogram of these averages.

Exercise 4.4. (*a*) Produce a time series of N steps of a 1-dimensional random walk with a variance of S. Produce a plot. (*b*) Produce a time series of N steps of a 2-dimensional random walk. Use a variance of S for both dimensions and a correlation of 0. Produce a 2D and a 3D plot.

Exercise 4.5 (The Sieve of Eratosthenes)**.** Implement the Sieve of Eratosthenes to find all prime numbers that are smaller than N.

Exercise 4.6. (*a*) Calculate the sum of all squared integers up to N. (*b*) Calculate the sum of all squared integers up to a maximum value of the sum of M. Display the last integer that contributed to the sum.

Exercise 4.7 (Magic square)**.** A magic square of size $n \times n$ has the following properties: (1) it contains all integers from 1 to n^2 exactly once, (2) the sum of all rows, columns and diagonals are equal. Verify that MATLAB's command `magic(n)` really produces a magic square for $n = 5$.

Exercise 4.8. (a) Write a function `dinterest`, that calculates the payoff of 1 dollar with anually discretely compounded interest. The function should have the same inputs and outputs as `ccinterest` from page 31. Start with a version for scalar inputs. Test your program with the following input: `dinterest(0.05, 5.5)`. (b) Extend your function `dinterest` to take vector input in either r or t. (c) Produce a plot of the payoff of one dollar at 10% interest per year, with continuous and discrete compounding, for maturities of one to 20 years, in steps of half years.

Exercise 4.9. Implement at least three different ways of a list of the numbers between 1 and 1000 that can be divided by both 7 and 13.

Exercise 4.10. Take the example of the nested for loops on page 45. We want to produce the same vector PV() as before, but calculated in a different order. (a) Change the order of the loops, so that the loop over the payment periods (k) becomes the outer loop and the loop over the rates (s) becomes the inner loop. *Hint:* You will possibly have to change and/or rearrange a few lines of code. (b) Which of the two ways that produce the same result is better? Write a short paragraph as a comment to your code discussing this question.

5. Working with data

References: Wooldridge, section 19.3

5.1. Introduction

If an economist wants to model and explain reality, data is the gold standard of his or her endeavor. An entire scientific field – econometrics – has evolved in order to make the best use of data that is often faulty, incomplete, censored or based on surveys at which people do not tell the truth. An industry has evolved to provide economists with data, sometimes free, as from the Federal Reserve or the ECB, sometimes costing more than $100'000. Modern computing equipment is capable of processing huge amounts of data, such as the 1 terabyte of tick data per year, that is produced at the US stock exchanges.

Whenever we use data, we have to keep in mind that it has been recorded, processed, transferred and converted by others. These steps produce inevitable errors. We should never blindly trust data, even if it has been obtained for a lot of money from a renowned source. The most important tool when working with data remains common sense.

Defintions

Data Observed/measured quantity of the study object. Usually varies with time and from individual to individual. Data can be discerned into *categorical*, *discrete* and *continuous* data.

In many econometric procedures, one part of the data (the "causes", policy variables or generally *explanatory* variables) are used to explain the other part (the "effects", outcomes or generally *explained* variables). Notation: x, x_i, x_t, X for explanatory variables and y, y_i, y_t, Y for the explained variables. In MATLAB, it is OK to call these variables simply `X` and `Y` if their meaning is clear.

Statistic A statistic is a function of the data, for example the following quantities: mean, median, mode, variance, AR(1) coefficient. Note that only few statistics are *model-free*, which means that most of them have to be interpreted in the context of the underlying model. This implies that our inference can be completely misguided if it is based on a statistic that has been calculated using a mis-specified model.

Parameter Quantity that specifies a certain property of a *model.* Parameter estimation is the process of finding the parameter that fits the data "best" according to a specific criterion. Notation: often Greek letters such as θ (for the parameter). Estimated parameters are often denoted with a "hat", e.g. $\hat{\theta}$. In MATLAB it is good practice to end the names of variables that describe estimated parameters with "-hat", e.g. `sigmahat`.

Fitted data A model can produce model-implied values for a set explained variables given explanatory variables and parameter estimates. More formally, a model can produce a conditional expectation for the observable quantity y given x, short $\hat{y} = f(\hat{\theta}, x)$. It makes no sense to write $\hat{x}$. Similar to estimated parameters, the name MATLAB variables containing fitted data with "-hat".

Envelope information. Describes the exact circumstances under which a data set has been obtained, i.e. the source of the original data, the selection and data cleaning procedure and the exact model specification for parameters and fitted data. It is good MATLAB practice to add envelope information in a structure whenever saving data (see page 191). Envelope information usually consists of:

- A reference to the input dataset (avoid duplication, do not save the input data)
- A reference to the MATLAB program (date/version) that has created this data
- The input parameters (if any), the specification of the data subset (if any) or the model specification
- The (estimated) parameters that have been used to create fitted data

5.2. Organizing data

The audit trail is the gold standard of empirical research: as soon as we have a result, we have to be able to show how it had been obtained, in every little step from the data to the final number. This means we have to be able to document exactly where our data has come from, how it has been processed (e.g. in sub-samples) and all subsequent calculations. An audit trail also includes code, choice of (sub) model, constraints, starting values, and assumptions. I would even go so far as saving the seed of the random number generator for full replicability.

Creating a data structure

There is no simple solution for creating a data structure. Depending on the problem and the nature of the data, different structures may be suitable. A few principles can help in designing a data structure:

- Store data in its most natural format (usually how it is observed)

- Whenever possible, store in matrix format. It is easiest to access and process.
- Avoid duplication. When a data set is used several times, its is better to reference it and not duplicate it.
- Solve the problem of missing data. There will always be missing data. You delete observations with missing data or interpolate to fill gaps. Both solutions have disadvantages.

MATLAB storage structures

- The simplest MATLAB data structure is the matrix, using the following conventions
  - *Time series data.* One day = one row, one individual (person, firm, stock, country) = one column, i.e. time series are always column vectors.
  - *Cross-sectional data.* One individual = one row.
  - *Panel data.* One individual = one column, one day = one row.
- The `dataset()` function is useful for working with cross-sectional data, see page 25.
- Store data with more than two dimensions (e.g. time series of covariance matrices) in an array (see page 25)
- For non-rectangular time series data use cell arrays (see example below).
- Time series data: add a vector of timestamps. Be careful when joining two time series (see exercise 5.3).
- The `Time Series Objects` and `Time Series Collection Object` combine time series data with the according time stamp. See the PC lab for an example.

Example: cell arrays to organize tick data

```
day1=0.01*randn(100,1)  Day one: 100 observations
day2=0.02*randn(150,1)  Day one: 150 observations
[day1 day2]             Does not work ...
ticks{1}=day1;          Start a cell array ...
ticks{2}=day2
plot(cumsum(ticks{1}))  Retrieve data of first day
for d=1:2               Loop ...
  rv(d)=var(ticks{d});  .. Calculate daily realized variance
end
ticks{1}(1:10)          Even this is possible: first 10 observations of day 1
```

NOTE: Should output data such as parameter estimates or fitted data be saved or recalculated every time? Saving and later re-loading data can introduce inconsistencies, when your saved results are based on an outdated version of your model or your parameters. A rule of thumb is to save a calculation output only if it is very costly to obtain (i.e. if it takes long to calculate).

Working with calendar dates

MATLAB provides useful commands for working with calendar dates, such as: `datenum()`, `datestr()`, `datevec()`, `weekday()`. The best way is to convert a date from any text format (such as "2012-02-02" or "Feb 12, 2012" into a time stamp. These is simply the number of days that have passed since January 1 in the year zero. Times of the day can be stored as fractional time stamps. Once dates are in the time stamp using datenum(), they can be easily added and subtracted. A useful command for plots is `datetick()`.

Example: working with calendar dates

`datestr(0)`	The start of all time stamps
`mydate='2012-09-29'`	Date as a string.
`d=datenum(mydate)`	Unix date number.
`datestr(d)`	Convert back to a date.
`datestr(d+0.75)`	Fractional time stamps correspond to times of the day.
`datestr(d+86)`	Which day is 86 days after this day?
`peak=datenum('2000-03-10')`	The day of the NASDAQ peak.
`weekday(peak)`	On which weekday was it?

5.3. Processing data

5.3.1. Preparing a data set

Load and convert data

MATLAB provides several commands for loading data, each with its advantages and disadvantages. It is possible to load data manually before starting a program, but it is preferable perform the data loading inside the program to ensure consistency between the runs of a program (everything that is done manually is a source of errors). Commands for loading data are: `xlsread()`, `csvread()`, `importdata()` and `textscan()`, the latter being the most powerful command.

After loading, data should be immediately converted if necessary (i.e. change of units).

Clean data

An (incomplete) checklist for cleaning data:

- Locate and read the description that came with the data
- Check measurement units (percentage/fractions, time in years/trading days/calendar days) and convert if necessary
- Check for completeness (every individual on every day in the time series)

- Find placeholders for “no data available” (often 0 or −1 or 99.99, read description). Replace by `NaN`.
- Think of what to do with non-available data. Is the econometric procedure capable of handling `NaN`? Will eliminating observations with `NaNs` introduce a bias?
- Verify selected entries form alternative sources
- Plausibility check 1: Range, summary statistics, extreme values
- Plausibility check 2: Innovations (first differences – most time series are persistent). Watch for single (jumps) and double (outliers) spikes.
- Cross check multiple datasets: many economic quantities are correlated.
- Economic check: arbitrage relations, natural ranges (e.g. interest rates > 0)
- Elimination of “outliers” or data that carries little information

Merge data

Merging data is an everyday procedure, even the simple calculation of an excess return requires us to merge stock and interest rate data. The following points should be observed when merging data:

- Verify that the data is *commensurate*, i.e. it has a common unit of measure (for example, all in percentage points or fractions of 1).
- Verify that the time series have common dates, see exercise 5.3.
- Use data structures for merging data of different dimensions.

NOTE: Cleaning and merging are sometimes interrelated. Cleaning data, for example, may require a merged data set if we want to exclude days on which there have not been observations in all relevant markets.

Calculate derived quantities

Derived quantities like returns or implied volatilities should only be calculated once a clean and merged data set has been produced.

Save data

Data is best stored locally in a MATLAB data file.An alternative comfortable solution for storing data for further use is a (local) database, see section 15.

TIP: Useful commands:
- `sum(X)`, `min(X)`, `max(X)`, `mean(X)`, `sort(X)`
- `cumsum()`, `diff()`, `prod()`, `cumprod()`, `sort()`, `sortrows()`, `sign()`

- Set operations: `unique()`, `intersect()`, `any()`, `all()`
- Calculations with relational operators, see page 38
- Special conversion functions: `x2mdate()`

5.3.2. Creating sub-samples

Sub-samples are widely used in financial econometrics for model validation. Depending on the purpose, different techniques are used.

In-sample v. out-of-sample

In order to verfiy whether a certain (usually parametric) model actually describes the underlying phenomenon and does not simply (over-)fit the data, many models are tested out of sample. The original data set is split into an “in-sample” period (usually the first half of the data), which is used to estimate the model’s parameters. Then the model is applied to the remaining part of the data (“out of sample”) and it is verified how much the model fits the data better in sample than out of sample. A large difference between the two is an indication of model misspecification and/or overfitting.

Random sub-samples

Random sub-samples are used if the amount of data is too large for the envisaged estimation procedure and if a fraction (e.g. one tenth) of the observations are sufficient to obtain a decent estimate. We randomly select a fraction of the observation, which means we can apply all the econometric theory that is based on random sampling.

Subsampling

Sub-sampling (not to be confronted with the creation of a sub-sample) means lowering the sampling frequency of a time series while maintaining its length, e.g. moving from daily data to weekly data. This is usually done by dropping the surplus observations. The opposite process, i.e. increasing the sampling frequency, entails some form of interpolation and is discussed on page 100. The following example shows random subsampling.

Example 6. The calculation of the realised variance should be independent of the sampling frequency. In fact, a random sub-sample should yield the same variance estimate (albeit less efficient). One way to study this is the creation of sub-samples, where randomly chosen observations are dropped. (It is important that the remaining observations remain in the same order). Imagine you have a function `RealizedVariance()` that calculates the realised variance over a vector of price observations. We then can use logical indexing to create sub-samples of the price data.

***Example:** Random sub-samples with logical indexing

```
T=size(myTick,1);              number of observations in the tick data
rand(10,1)>0.5                 this creates a vector of zeros and ones with ca. 50% ones
rand(10,1)>0.9                 only ca. 10% ones
size(myTick(rand(T,1)>0.5))    sub-sample with ca. 50% of observations
size(myTick(rand(T,1)>0.9))    sub-sample with ca. 10% of observations
myTick(rand(T,1)>0.9)
RealizedVariance(myTick)       Reference value
RealizedVariance(myTick(rand(T,1)>0.5))
                               repeat this several times
for k=1:1000
  myVariance(k)=RealizedVariance(myTick(rand(T,1)>0.5));
end
hist(myVariance)               Distribution of realised variances in the sub-samples
```

5.4. Data sources

Free data resources

- Financial data
  - `finance.yahoo.com`. Free financial data (click on "Historical Prices" when viewing a ticker). Use `sqq()` to access the data, available at `www.mathworks.com/matlabcentral/fileexchange/4069-stock-quote-query`
- Macro data
  - `research.stlouisfed.org/fred2`. Use `fred2read()` to access the data, available at `www.mathworks.com/matlabcentral/fileexchange/40912-fred2read`
  - Further central bank data `www.newyorkfed.org`, `www.ecb.int/stats`, `www.bundesbank.de/statistik`, `www.snb.ch/en/iabout/stat`
  - `data.worldbank.org`, use `getWbData()` to load the data directly into MATLAB, available at `www.kimjruhl.com/computing`
  - `www.freelunch.com`
- Government open data initiatives
  - `www.data.gov`
- Other data collections
  - `www.quandl.com` A portal for economics and finance data, with MATLAB interface. You need to register to be able to use the MATLAB interface. See PC lab.

Commercial data providers

- `www.crbtrader.com`
  Low-cost financial data, including futures and options, mostly US.

5.5. PC lab: Importing data

- Go to `finance.yahoo.com`, select the S&P500 index, download historical data
- Load the data into MATLAB using the interactive mode
- Produce a plot of the data.

Using the sqq package

- Download the `sqq` package from `matlabcentral.com`
- Install the `sqq` package by adding it to the MATLAB path (see page 190, "your personal toolbox.")

```
[date, close]=sqq('MSFT','2004-01-01',today)
```

`plot(close);`	Stock chart.
`plot(date,close)`	Nicer ...
`datetick('x',10)`	Read the documentation of `datetick`

5.5.1. Working with quandl.com

Quandl is a free and powerful web repository for all sorts of numerical data, including financial data. It provides direkt links ("APIs") to many programming languages, including MATLAB. If you use the quandl API for more than 50 calls per day, you need a free registration.

Installing the quandl software

1. Go to `http://www.quandl.com/help/matlab`
2. Follow the information to download the MATLAB quandl package. Unzip it.
3. Save the `+Quandl` folder in a convenient location and add the parent folder to the MATLAB path (see page 190). Do not add the `+Quandl` folder itself to the path or change its name.
4. Start MATLAB and try the following commands:

`myDat = Quandl.get('QUANDL/EURCHF')`	Result is a time series collection object.
`plot(myDat.Rates)`	The time info is automatically added.
`myDat.TimeInfo.Format = 'yy';`	This formats the x-axis in a better way
`plot(myDat.Rates)`	Plot again to see the result.

Registration and authentication (optional)

1. Go to `www.quandl.com`

2. Click on “Sign up”. Registration is free.
3. Fill in the form
4. Once registered, log in and go to “Account” under your user name
5. Find the “API key”, a long combination of small and large letters. This is your “token”
6. In MATLAB authenticate with `Quandl.auth('your token');` (Only done once.)

Useful quandl database codes
To find a database code, search in the data browser, then locate the “permanent link” at the lower left part of the screen and use the part of the link after `https://www.quandl.com/`

- *Exchange rates:* `QUANDL/EURCHF`, `BAVERAGE/USD` (Bitcoin)
- *Stocks:* `GOOG/NYSE_MMM` (for NYSE stocks), `GOOG/NASDAQ_AAPL` (for Nasdaq stocks)
- *Futures:* `CHRIS/CME_ED1` (3 month eurodollar), `CHRIS/CBOE_VX1` (VIX front month), `CHRIS/EUREX_FDAX1` (DAX), `CHRIS/EUREX_FGBL1` (Bund future)
- *Macro:* `WORLDBANK/DEU_GFDD_OE_02` (German CPI), `WORLDBANK/USA_NYGDPMKTPXN` (US GDP)

Some empirical work using quandl
See `demo05.m`

5.6. PC lab: Identifying US recessions

Go to the website of the “Federal Reserve Economic Data” at `http://research.stlouisfed.org/fred2/categories`. Select *National Income & Product Accounts* and then *GDP/GNP* and finally the series “Real Gross Domestic Product, 3 Decimal”. Read the description. Download the function `fred2read()` from `http://www.mathworks.com/matlabcentral/fileexchange/40912-fred2read` and add the function to the MATLAB path.

In this simple example, we now want to identify recessions, which we define as two or more consecutive quarters of GDP drop. We do this by looking back two quarters. This means we can only start at quarter number three.

Load the data

```
[calDate GDP header]=fred2read('GDPC1');
size(GDP);                          How many observations do we have?
GDP(1:10)                           Print first few observations
datestr(calDate(1:10))              Print the calendar dates of the first few observations.
disp(header)                        Show the envelope information.
```

Indentify recessions

```
recess=zeros(length(GDP),1);        Initialize recessions vector ... all zeros.
for t=3:length(GDP)                 Start loop at Q3 → minimum to look back 2 quarters
  if GDP(t)<GDP(t-1) && GDP(t-1)<GDP(t-2)  Drop in this and the previous quarter
     recess(t)=1;                          Then this quarter is in a recession ..
    recess(t-1)=1;                          ... and the previous one.
  end
end
sum(recess)                         Number of quarters in recession
mean(recess)                        Fraction of recession quarters
```

Make a plot

```
figure                              Start a nice plot with grey bars for recessions
bar(calDate,recess*16000,'EdgeColor','none','FaceColor',[0.8 0.8 0.8],'BarWidth',1);
hold on
plot(calDate,GDP)                   Plot GDP over the bars
datetick('x',10)                    Change x-axis scale to calendar years
```

Compare result to the official NBER recession indicator

```
[calDateNBER recessNBER header]=fred2read('USRECQ');
bar(calDateNBER,recessNBER*8000,'EdgeColor','none','FaceColor',[0.5 0.2 0.2],'BarWidth',1);
axis([calDate(1) calDate(end) 0 16000])
                                    The last line limits the display to 1947 and on, as USRECQ
                                    has a longer history than GDPC
```

5.7. Exercises

Exercise 5.1. Go back to the demo program on page 50. Write a program that produces 20 days of tick data. The number of ticks per day should be a random number between 100 and 200. On each day, the standard deviation of the simulated returns should be randomly drawn between 1% and 5%. Now

(*a*) retrieve the first 5 returns of day 10 in the sample, (*b*) retrieve the last 5 returns of day 3 in the sample, (*c*) retrieve the last return of the last day in the sample, (*d*) produce a plot of the estimated daily realized variances, (*e*) produce one plot showing the (overlapping) stock price movements on day 1,10 and 20.

Exercise 5.2. Use `sqq` to load the stock prices of Apple Computer (AAPL) and Microsoft (MSFT) from January 1, 1999 to yesterday. (*a*) Looking back over such a long sample, how large was the realized variance of returns of holding one stock of AAPL viz. MSFT? *Hint:* this exercise is not as simple as it seems. Inspect the data carefully to spot any problems. (*b*) Create weekly and monthly returns for AAPL and MSFT. To create the weekly data, start from the daily data set and use Wednesdays if available, if not use the Tuesday of the same week and if this is also not available, the Thursday. To create the monthly data, start again form the daily data set and use the last day of the month for which data is available. *Note:* Start from the daily data set that you have retrieved using `sqq`. As always, do not use any toolbox functions. (*c*) Calculate the realized variance based on daily, weekly and monthly data. Which variance do you expect to be larger? Are you surprised? (Answer these questions in one sentence as a comment in your program).

Exercise 5.3. Go to `finance.yahoo.com` and find the ticker symbols that yahoo uses for the S&P 500 index and the DAX. Use `sqq` to retrieve the values of these indices for the period from January 1, 1999 until yesterday. (*a*) produce a joint time series of DAX and SP500 for those days only on which both indices were traded. To do this, write a function `commontimeseries` that takes two (or better N) sets of dates and values and produces one common time series. (*b*) calculate the standard deviation and the correlation separately for every calendar year in your sample. Produce a plot of the time series of yearly standard deviations and correlations that you obtain. (*c*) What did you learn about international diversification? (Answer this question in one sentence as a comment in your program). *Note:* Do not use any toolbox functions and do not use MATLAB's time series objects.

Exercise 5.4. *Go back to the GDP/recessions par of the PC lab. Given the variable `GDP`, the whole identification of recessions could be done in two lines without a loop. How?

Exercise 5.5 (Moving averages)**.** Use `sqq()` and download the daily share prices for Apple Computer (AAPL) from 2000-01-01 to today. Make a sensible decision which data series you use. (*a*) Calculate the daily returns, make a plot and a histogram. (*b*) Calculate the 22-day moving average (22 trading days) and plot both the share price and the moving average in one plot. Do not use any toolbox functions. (*c*)$*$ Calculate the 30-day moving average (30 *calendar* days) and make the same plot as in (b).

Exercise 5.6 (Up to you.)**.** The list of data sources in Section 5.4 is only a glimpse of the data that is available on the web. Find at least ten reputable data sources (usually web sites), if possible with MATLAB integration.

6. Linear algebra and its applications

References: LeSage, Creel, Verbeek, Greene

Linear matrix algebra is an excellent tool for applied economics, as a matrix is the natural representation of data. MATLAB is very good at handling matrices, as the matrix is the native data type in MATLAB. This section showcases some applications of matrix algebra in economics, econometrics and finance.

6.1. Solving systems of linear equations

Take a system of linear equations, like

$$\begin{aligned} 2x + 3y + 4z &= 23 \\ 2x - 1y + 2z &= 8 \\ 3x + 4y + 5z &= 30 \end{aligned} \tag{6.1}$$

This system can be written in matrix form as

$$\begin{pmatrix} 2 & 3 & 4 \\ 2 & -1 & 2 \\ 3 & 4 & 5 \end{pmatrix} \begin{pmatrix} x \\ y \\ z \end{pmatrix} = \begin{pmatrix} 23 \\ 8 \\ 30 \end{pmatrix} \tag{6.2}$$

Or shorter as

$$A\mathbf{x} = b \tag{6.3}$$

The solution is then

$$\mathbf{x} = A^{-1}b \tag{6.4}$$

Solving a system of equations

`A=[2 3 4; 2 -1 2; 3 4 5]`	
`b=[23 8 30].'`	Column vector
`x=inv(A)*b`	One possible solution
`x=A\b`	Alternative with Gaussian elimination
`A*x`	Verify result
`help mldivide`	Find out more about the matrix division using \
`A=[2 3 4; 2 3 4; 3 4 5]`	
`x=inv(A)*b`	A has two linear dependent row vectors → error.
`rank(A)`	A does not have full rank.

6.2. PC-Lab: From equation to program – OLS step-by-step

Manually programming an estimator like OLS seems to be unpractical, but we perform this task for two reasons. First, we can watch the estimation process step-by-step, which might help understanding it. Second,by programming a known and simple estimator, we prepare for implementing more difficult ones.

Data preparation

`load attend.txt`	Load data.
`[n,k]=size(attend)`	Number of rows, number of columns.
`type attend.des`	Check the data description and decide which part of the data to use.
`Y=attend(:,2);`	New explained (Y) …and explanatory (X) variables.
`X=[ones(n,1) attend(:,1) attend(:,3) attend(:,7) ]`	
`[n,k]=size(Y)`	Verify the dimensions of the sample that we are going to use.
`[n,k]=size(X)`	

Verifying the assumptions

MLR.1 can be verified by inspecting the regression equation. **MLR.2** and **MLR.3** cannot be verified. To verify **MLR.4**, calculate the rank of X using `rank(X)`. See Appendix C.

OLS estimate

`rank(X)`	Before we start, check **MLR.4**
`rank(X.'*X)`	This is the same by construction: $rk(A) = rk(A'A)$
`b=inv(X.'*X)*X.'*Y`	The regression coefficients (we write `b` for β).
`u=X*b-Y`	The residuals (we write `u` for ε.)
`mean(u)`	Not a verification of **MLR.2**. $\sum \varepsilon_i = 0$ by construction of the estimator.

Error terms and R^2

`SST=(Y-mean(Y)).'*(Y-mean(Y))`	This is equivalent to $SST = \sum_{i=1}^{n}(y_i - \bar{y})^2$.
`SSR=u.'*u`	This is equivalent to $SSR = \sum_{i=1}^{n} \varepsilon_i^2$.
`R2=1-SSR/SST`	$R^2 = 1 - SSR/SST$
`si2hat=u.'*u/(n-k-1)`	Estimate $\hat{\sigma}^2$.
`sihat=sqrt(si2hat)`	And calculate $\hat{\sigma}$.

Standard errors and t-test

`Xinv=inv(X.'*X)`	First step towards standard errors.
`dXi=diag(Xinv)`	New command `diag`, self-explaining.
`sdXi=sqrt(dXi)`	Vector of $\sqrt{((X.'X)^{-1})_{jj}}$
`seb=sihat*sdXi`	$se(\hat{\beta}_j)) = \sigma\sqrt{((X.'X)^{-1})_{jj}}$
`tval=tinv(.95,n-k-1)`	Value of the t-statistic for 90% confidence interval. (Admittedly, `tinv()` is used from the statistics toolbox.)
`bmin=b-tval*seb`	Lower bound.
`bmax=b+tval*seb`	Upper bound.
`[b seb]`	Output β and its standard erros.
`[bmin b bmax]`	Output the confidence interval and the point estimate of β.
`tstat=b./seb`	Values for the t-statistic. Note the dotted (.) division operator.
`pval=2*tcdf(-abs(tstat),n-k-2)`	$n-k-2$ degrees of freedom, because we did not count the constant in k.
`[b seb tstat pval]`	The p-values and a typical output.

6.3. PC-Lab: The MATLAB Statistics Toolbox

MATLAB has a very limited number of built-in statistics functions. Many are only available in the Statistics toolbox, which is included in the student license. Serious econometrics, however, can only be done by manually programming or by using the Econometrics toolbox (see next section).

`diary('statistics.txt')`	Don't forget to start a diary.

Simple estimate: regress()

If you have not yet done so, perform the data preparation steps from the previous PC lab.

`b=regress(Y,X)`	Simple regression.
`mean(X)`	The means of each observable in X.
`mean(Y)`	The mean grade.
`mean(X)*b`	This is the same as `mean(Y)`. Why?
`[b, biv]=regress(Y,X)`	95% confidence intervals for β.
`[biv(:,1) b biv(:,2)]`	Lower bound, point estimate, upper bound for each component of β.

More detailed statistics: regstats()

The MATLAB command `regstats()` is part of the Statistics toolbox. It produces a wide range of statistics. Its use is different from all other commands, as `regstats()` automatically adds a column of ones. Thus, we need to define a different data matrix first.

```
X1=[attend(:,1) attend(:,3) attend(:,7) ]
stats = regstats(Y,X1,'linear')
```

The resulting `stats` is a data structure. To access the individual contents, use a dot.

```
stats.beta        Regression coefficients
stats.covb        Covariance of regression coefficients
stats.yhat        Fitted values of the response data
stats.r           Residuals
stats.mse         Mean squared error
stats.rsquare     R-square statistic
stats.adjrsquare  Adjusted R-square statistic
stats.tstat       t statistics for coefficients
stats.fstat       F statistic
```

Note that a few components of `stats` are structures, again, for example `tstat`. Example: to access the p-value of the t-statistics, use `stats.tstat.pval`

6.4. PC-lab: applications of spectral theory[1]

`diary('spectral.txt')`	Don't forget to start a diary.

Eigenvalues and rank

`A=[2 3 4; 3 -1 2; 4 2 2]`	Symmetric matrix
`B=[2 3 4; 2 3 4; 3 4 5]`	
`rank(A),rank(B)`	One matrix has full rank, one not.
`eig(A)`	Vector with eigenvalues.
`eig(B)`	One or more zero eigenvalues $\leftrightarrow$ not full rank

[1]See appendix A for the theory.

Matrix diagonalization

`[P, D] = eig(A)`	Eigenvectors and eigenvalues.
`dot(P(:,1),P(:,2))`	Columns of `P` are eigenvectors.
	They are orthogonal, because A is symmetric.
`P*D*inv(P)`	Reproduce `A` from the diagonalization.
`D`	Some eigenvalues are negative.
`sqA=P*sqrt(D)*inv(P)`	An application: the square root of a matrix.
`sqA*sqA`	Verify: this is equal to A.

Choleski decomposition and correlated random numbers

Try to simulate a portfolio of two stocks $i = 1, 2$. Each stock return is normally distributed with mean μ_i and variance σ_i^2, but it is further known that the returns are correlated with a correlation coefficient ρ. For simplicity, $\mu_i = 0$ and $\sigma_i^2 = 1$.

Naive version

`z1=randn(1000,1);`	$Z_1 \sim N(0,1)$
`z2=randn(1000,1);`	Z_2 ... same
`rho=0.5;`	Choose some number for ρ.
`x1=z1;`	
`x2=rho*z1+(1-rho)*z2;`	
`corr(x1,x2)`	Big surprise.

What went wrong? With $\mathbf{x} = Q\mathbf{z}$ we can calculate the variance:
$V(\mathbf{x}) = E[\mathbf{x}\mathbf{x}'] = E[Q\mathbf{z}\mathbf{z}'Q'] = QE[\mathbf{z}\mathbf{z}']Q' = QQ'$

Correct version

`z = randn(1000,2);`	Produce z_1 and z_2 in one step.
`Sigma=[1 0.5; 0.5 1];`	Desired variance-covariance matrix.
`R = chol(Sigma);`	Perform a Choleski decomposition.
`x = z*R;`	Produce correctly correlated random numbers.
`corr(x)`	Now we get what we expected.

The result does not perfectly match the desired covariance matrix, because the random numbers produced by MATLAB are slightly correlated. We can improve the result by taking out the initial correlations first.

`cor=[1    0.2  0.8;`	Desired correlation matrix
`     0.2  1   -0.2;`	
`     0.8 -0.2  1  ];`	
`X0=randn(1000,3);`	Random numbers produced by MATLAB ...
`corr(X0)`	... are not perfectly uncorrelated.
`B=chol(inv(corr(X0)));`	To take out initial correlations ...
`X0better=X0*B;`	... multiply by Choleski decomposition of inverse.
`corr(X0better)`	Much closer to unity.
`A=chol(corr);`	Second Choleski decomposotion introduces correlations
`Xsimple=X0*A;`	Correlated random numbers based on simple version
`Xbetter=X0better*A;`	Correlated random numbers based on improved version
`corr(Xsimple)`	Simple correlated numbers are quite far from ...
`corr(Xbetter)`	... the desired correlation matrix.
`diary off`	Now it is time to mail yourself your diary.

SVD decomposition

Download the program `svd_foto.m` and the image `nash.jpg` and run the program in cell mode. Change the value of `max` in line 22 to different values between 10 and 300.

6.5. Exercises

Exercise 6.1. Calculate using Matlab the eigenvalues and eigenvectors of the following matrices. Verify equation A.4. Comment on the result.

(*a*) `A=magic(5)` (*b*) $B = \begin{pmatrix} 1 & 1 \\ 0 & 1 \end{pmatrix}$

Exercise 6.2. Download `svd_manual.m`, read the code and try to understand it.

Exercise 6.3. (*a*) Compute the rank of at least ten random matrices of different size (square and non-square). Can you find a rule? (*b*) Compute the eigenvalues of several large square matrices created by `rand()`. Find a convenient graphical representation. Is there any pattern?

Exercise 6.4. X contains a data set with the three variables $EXPR$, $MALE$ and $FEMALE$. $EXPR$ stands for years of work experience, $MALE$ is a dummy variable that is 1 if the individual is male and $FEMALE$ is a dummy variable that is 1 is the individual is male.

Let $x = \begin{bmatrix} 5 & 0 & 1 \\ 7 & 1 & 0 \\ 12 & 0 & 1 \\ 3 & 1 & 0 \end{bmatrix}$.

(a) calculate the rank of X.
(b) create three new matrices A, B and C that contain pairwise two of the three variables. Calculate the rank of each of these matrices.

Exercise 6.5. There is no *vec*() function in MATLAB. Write a short program that calculates $vec(A)$ for a matrix A of any size. Test your program with the matrix `A=magic(3)`. *Hint:* this exercise can be implemented in at least three different ways. One obvious way is to use a `for`-loop (which has not yet been discussed), but there are alternatives. One is a smart way to use the colon operator (explained in `help colon`), another one is to use the command `reshape()` (see help).

Exercise 6.6. (a) Why did we only take four columns of `attend` when we declared `X=[ones(n,1) attend(:,1) attend(:,3) attend(:,7) ]`? (b) Show with MATLAB, that taking the all columns of `attend` would not work. (c) Find the maximum number of variables of `attend`, for which we can still perform an OLS regression. *Hint:* there is a very convenient way of finding linearly dependent variables in X. Try to use this method. If you cannot find it, you can use trial and error. (d) Repeat the PC-Lab with the new, larger data set. (e) If possible, compare the results. Did anything change?

Exercise 6.7. Take X and Y from the previous exercise. (a) Perform a full regression of Y on X, including t-statistics. (b) Change the scale of `'homework'` such that it ranges from 0.125 to 1 instead of [12.5, 100] (i.e. not percentage points but fraction of total homeworks). Repeat the regression. How do the β coefficients change? Does the significance change, as well? (c) What do we learn from this exercise about interpreting β-values?

Exercise 6.8. Start with `X=[ones(n,1) attend(:,1) attend(:,3) attend(:,7) ];` and `Y=attend(:,2)`. (a) create five variables called `X1` through `X5`, that contain the first, second, ... hundred observations of `X`. (So each X_i is different and each contains 100 observations). Create the corresponding variables `Y1` through `Y5`. (b) Perform an OLS regression of `Y1` on `X1`, `Y2` on `X2` and so on. Compare the results and make a comment. (c) Would it make sense to regress `Y1` on `X2` and so on?

Exercise 6.9. A simple extension to OLS is to discard assumption **MLR.5**. Heteroskedasticity robust errors

$$Var(\hat{\beta}) = (X'X)^{-1}(X'\hat{\Sigma}X)(X'X)^{-1}$$

with $\hat{\Sigma}$ a diagonal matrix with $\hat{\Sigma}_{ii} = \hat{\epsilon}_i^2$. Repeat the PC-Lab on manual estimation with these new error term definition. Use the `diary` command to document your work.

Exercise 6.10 (Up to you.)**.** Find 3 concrete applications of linear algebra in your field. (Examples and exercises in this chapter do not count).

7. Simulation-based methods

References: Greene, Appendix E.3; Brandimarte chapters 4 and 7; Judd, section 8, Glasserman, Gilli, Maringer and Schumann, sec 6, 8

7.1. The usefulness of random numbers

The basis for every simulation is a model: a coin delivers heads with probability p and tails with probability $1-p$. If the coin is a fair coin, p is 0.5. A fair dice has a 1/6 probability for every number. Once we have a model for the behavior of a coin, a dice, an economy or a financial market, it is an obvious step to simulate it.

Example 7 (Rolling a dice in MATLAB). MATLAB's `rand()` function produces uniformly distributed random numbers between zero and one. For our dice, we need a discrete distribution of integers from one to six. We can obtain this via a two step transformation: First, we multiply by 6, which gives a uniformly distributed random number between zero and 6. Second, rounding up to the next integer gives the desired discrete distribution. The mathematical notation is $x \to \lceil 6 \cdot x \rceil$. In MATLAB, rolling one dice looks like this:

Example: Rolling a dice in MATLAB

`dice=randi(6);`	Draw from discrete distribution
`dice=ceil(6*rand);`	Draw from uniform distribution and transform

NOTE: A common error is to use `round()` instead of `ceil()`. Try `round(6*rand)` several times to understand this error.

7.1.1. Producing random numbers in MATLAB

MATLAB – just as any other computer system – cannot produce truly random numbers. It rather produces a deterministic sequence of numbers that have almost no correlation, which is why they are called *pseudo-random numbers*. When MATLAB is started, the random number generator is initialized and will always produce exactly the same deterministic series. For some applications (e.g. to repeat a simulation experiment) this is useful, in other cases (e.g. to enlarge a Monte Carlo sample) this must be absolutely avoided.

Distribution	probability fn. p.d.f.	MATLAB command	See ...
Bernoulli 0/1	$P(0) = P(1) = \frac{1}{2}$	`round(rand)`	
Symmetric Bernoulli	$P(-1) = P(1) = \frac{1}{2}$	`sign(randn)`	
Bernoulli	$P(0) = q;\ P(1) = p$	`rand<=p`	
Binomial (n,p)	$P(k) = \left(\begin{array}{c} n \\ k \end{array} \right) p^k q^{n-k}$	`sum(rand(n,1)<=p)`	
Uniform $[0 \ldots 1]$	U(0,1)	`rand`	
Uniform $[a \ldots b]$	U(a,b)	`a+(b-a)*rand`	
Standard normal	$N(0,1)$	`randn`	
Normal	$N(\mu, \sigma^2)$	`sigma*randn+mu`	
Correlated n-dim. norm.	$\mathbf{x} \sim N(\mu, \Sigma)$	`mu+randn(n,1)*chol(Sigma)` or `mvnrnd(mu,sigma)`	p. 63

Table 7.1.: Random number generation in MATLAB.

To control the starting point of the sequence, one can manually set the *seed* of the random number generator.

Example: random number generation

```
myRand=rng               Obtain state of random number generator.
rand(1,5)                Produce five random numbers
rand(1,5)                Produce five different ones
rng(myRand);             Reset the random generator
rand(1,5)                Produce same five random numbers
rng(sum(100*clock))      Unique initialization.
```

Built-in distributions

MATLAB has three built-in distributions: the standard uniform distribution `rand()`, the standard normal distribution `randn()` and the uniform integer distribution `randi()`. For all other distributions, there are two options:

- Transform the basic distributions (see next two subsections)
- Use the `random('distribution',A,B,C)` command from the statistics toolbox, which supports more than 20 distributions. Depending on the type of distribution, up to three parameters `A,B,C` have to be provided.

General procedure to sample from discrete distributions

A discrete distribution takes the values $x_i (1 \leq i \leq n)$ with probabilities p_i. As usual, $p_i > 0$ and $\sum p_i = 1$. First we consider the simple case that $x_i = i$.

Example 8. We want to produce random draws from the distribution

p_i	0.2	0.1	0.4	0.3
x_i	1	2	3	4

To produce draws from this distribution, we start with uniformly distributed random numbers and imagine that the interval $[0, 1]$ is divided into intervals with the lengths $p(1), p(2), \ldots p(n)$. Our discretely distributed random number is the number of the interval in which the uniform random number lies.

```
p=[0.2 0.1 0.4 0.3];        small 'p': probabilities p(1) ... p(4)
P=cumsum(p)                 large 'P': cumulative probability (upper bracket values)
r=rand;                     Save random number for next two lines
sum(r>P)+1                  Discrete random value
r>P                         What did we do?
for k=1:1000                As a short verification, ...
  dist(k)=sum(rand>P)+1;      we produce 1000 random draws
end
hist(dist)                  And plot a histogram.
```

Example 9. For general values of x, we need one more step. Consider

p_i	0.2	0.1	0.4	0.3
x_i	-1.1	0	3.2	17

Note that `p` and `P` are the same as previously. To obtain our draws, we have to define a vector `x=[-1.1 0 3.2 17]`. We want to obtain `x(1)` with 20% probability, `x(2)` with 10% and so on:

```
x=[-1.1 0 3.2 17];          Define the vector of x-values
x(sum(r>P)+1)               A random draw
```

General procedure to sample from continuous distributions

The standard way to produce random numbers of any continuous distribution is to use the inverse function of the cdf: $G(y) = F^{-1}(x)$. If u is uniformly random distributed on $[0, 1]$, then $G(u)$ is distributed with distribution $F(\cdot)$.

Example 10 (Pareto distribution). The pdf of the pareto distribution is $f(x) = \frac{\alpha}{(1+x)^{1+\alpha}}$ for $x > 0$. The cdf is accordingly $F(x) = 1-(1+x)^{-\alpha}$. Its inverse is then $G(y) = (1-y)^{-\frac{1}{\alpha}} - 1$

We can also plot the CDF from is direct and from is inverse defintions (note that the argument `Xi` is used as y-values for the inverse CDF).

```
alpha=5;N=1000;
parCDF=@(x,alpha) 1-(1+x).^-alpha;
parICDF=@(x,alpha) (1-x).^(-1/alpha) - 1
Xi=linspace(0,1,N);
plot(Xi, parCDF(Xi,alpha) )
hold on
plot(parICDF(Xi,alpha), Xi, 'r--')
axis([0 3 0 1]);
```

We can now draw from the inverse pareto distribution

```
paretoDraw=@(alpha) parICDF(rand,alpha)
paretoDrawN=@(alpha,N) parICDF(rand(N,1),alpha)
```

The cdf of the Pareto distribution is alternatively obtained via simulation

```
myDraw=paretoDrawN(5,N)
plot(sort(myDraw),Xi)
```

7.2. Drawing from the empirical distribution

7.3. Simulating random processes

7.3.1. Simulating discrete-time random processes

A discrete-time random process is a series of random variables that are linked by a transition equation:

$$x_t = f(x_{t-1}) \qquad t \geq 1 \tag{7.1}$$

where $f(\cdot)$ is a random function. Simulating random processes is a major ingredient in calculating statistics of this process such as expectations or estimations of the process parameter. To do so, we need to know the functional form of $f(\cdot)$ and the value of x_0

Brownian motion. The simplest random process is the Brownian motion:

$$x_t = x_{t-1} + \varepsilon_t \qquad \varepsilon_t \sim N(\mu, \sigma^2) \tag{7.2}$$

where μ is called the drift of the process and σ^2 is the variance. If $\mu = 0$ and $\sigma = 1$, the process is called Standard Brownian motion. The Brownian motion process has no "memory", i.e. the probability distribution of tomorrow depends only on the state of today and not on the past history. This property is called the Markov property.

To simulate such a process, we need to make an assumption for x_0. Note that in MATLAB, `x(0)` does not exist. We need to use `x(1)` for x_0. This is a major source of errors.

```
x(1)=0;                   Staring value. We use x(1) for x_0, because there is no x(0).
for t=2:100               Loop over time, start with 2.
  x(t)=x(t-1)+randn;      Standard Brownian motion, ε_t ~ N(0,1)
end
```

Mean reversion: Ornstein-Uhlenbeck process.

$$x_{t+1} = x_t + \kappa(\theta - x_t) + \varepsilon_t \qquad \varepsilon_t \sim N(0, v) \tag{7.3}$$

```
% Three Ornstein-Uhlenbeck processes
kappa=0.04; theta=0; v=0.2;
x=[5;0;-5];
for t=2:150
  x(:,t)=x(:,t-1)+kappa.*(theta-x(:,t-1))+v.*randn(3,1);
end
plot(x.')
```

Stochastic volatility. Models with stochastic volatility, actually consist of two processes: one for the modelling qunatity (e.g. returns, interest rates) and one for the variance of above quantity. When simulating such processes, we usually have to simulate both processes step-by-step, i.e. first the variance process for t, then the main process for t before moving to $t + 1$.

Example 11 (GARCH.). The GARCH(1,1) model is usually denoted as

$$\begin{aligned} \ln(S_t) &= \ln(S_{t-1}) + \mu + \epsilon_t \\ \epsilon_t &= \sigma_t z_t \qquad\qquad z_t \sim N(0,1) \\ \sigma_t^2 &= \omega + \beta_1 \sigma_{t-1}^2 + \alpha_1 \epsilon_{t-1}^2 \end{aligned} \tag{7.4}$$

Note that the σ_t-process depends on ϵ_{t-1}. As we have to know ϵ_{t-1} before we can simulate σ_t, we have to simulate the process in a strict order: (1) σ_{t-1}, (2) ϵ_{t-1}, (3) σ_t, (4) ϵ_t and so on.

```
a1=0.2; b1=0.75;           % Set parameters
w=0.05;
epsilon(1)=0;              % Initialize epsilon-process.
```

```
sigma(1)=sqrt(w);
for t=2:100
  sigma(t)=sqrt(w + ...
      b1*sigma(t-1)^2 + ...
      a1*epsilon(t-1)^2);
  epsilon(t)=sigma(t)*randn;
end
```

```
sigma2(1)=w;
for t=2:100
  sigma2(t)= w + ...
      b1*sigma2(t-1) + ...
      a1*epsilon(t-1)^2;
  epsilon(t)=sqrt(sigma2(t))*randn;
end
```

```
S=exp(cumsum(epsilon));   % Convert from log returns to asset prices.
```

NOTE: There are two alternative ways to paramtetrize the GARCH process: using σ or σ^2 as a variabe. Have a look at the two alternatives above. Whenever possible, it is preferable to use the basic quantity as a variable. In this case, it is σ (`sigma`) in the left case as opposed to σ^2 (`sigma2`) in the right case. Whatever you choose, the most important thing is to stick to your convention.

Example 12 (Discrete-time Heston model.). The transition equations for the discrete-time Heston model (sometimes also called Feller diffusion) are:

$$\begin{aligned}
\ln(S_t) &= \ln(S_{t-1}) + \mu + \sigma_t w_t && w_t \sim N(0,1) \\
\Delta\sigma_t^2 &= \theta(\omega - \sigma_{t-1}^2) + \xi\sigma_{t-1} z_t && z_t \sim N(0,1) \\
\sigma_t^2 &= \sigma_{t-1}^2 + \theta(\omega - \sigma_{t-1}^2) + \xi\sigma_{t-1} z_t && \\
& cov(w_t, z_t) = \rho &&
\end{aligned} \tag{7.5}$$

Simulating this model is part of exercise 7.1.

7.3.2. Discretizing continuous-time processes

A continuous-time process must be discretized to a time scale of Δt before we can simulate it. The simples discretization scheme is the first order approximation or Euler approximation.

Process:	$dS_t = \mu(t)dt + \sigma_t(t)dW_t$
Basic idea:	$S_{t+dt} = S_t + \int_t^{t+dt} \mu(u)du + \int_t^{t+dt} \sigma(u)dW_u$
First term:	$\int_t^{t+dt} \mu(u)du \approx \mu(t)dt \qquad$ ("left point rule")
Second term:	$\int_t^{t+dt} \sigma(u)dW_u \approx \sigma(t) \int_t^{t+dt} dW_u = \sigma(t)(W_{t+dt} - W_t) = \sigma(t)\sqrt{dt}\, z_t$, $\quad z_t \sim N(0,1)$
Approximation:	$S_{t+1} = S_t + \mu_t \Delta t + \sigma_t \sqrt{\Delta t}\, z_t$
Rule of thumb:	$dt \to \Delta t; \qquad dB \to \sqrt{\Delta t}\, z_t.$

NOTE: When comparing discrete-time processes and their continuous-time cousins, note that Δt is often hidden in the discrete time parameters. This means we have to scale by Δt viz. $\sqrt{\Delta t}$ to compare parameters in the two notations.

Improved simulation schemes try to reduce the apprpoximation error for the above integrals, for example a second-order approximation (Milstein scheme) or, where possible, the exact simulation.

7.3.3. Some probability foundations

Probability measure $P(\cdot)$

A probability measure is a numeric value for every (measurable and relevant) state of nature. *Example:* When throwing coins, we assign +1 for heads and 0 for tails. This is measurable. Note that we do not regard the position, the speed or the color of the coin.

Sample X

A sample is a random draw from the population. Samples are **random**. This is important: it is the reason why all functions of the data will be random variables. The sample is what we call "data". *Examples:* $X_1 = [0, 1, 1, 0, 1]$; $X_2 = [1, 0, 1, 1, 1]$

Parametric model $\mathcal{M}$

A parametric model is an assumption about our population. A model is a member of a parametric family

$$\mathcal{M} = \{f(y, x, \vartheta) : \vartheta \in \Theta \subset \mathrm{R}^d\}$$

A model is described using the functional form $f(\cdot)$ and the parameter vector ϑ. Some parameters can only take certain values (e.g. $\sigma \geq 0$ for the normal distribution). This is expressed by the fact that $\vartheta \in \Theta$, where the latter is called parameter space.

Example 13 (Normal distribution). $f(x, \vartheta) = \frac{1}{\sigma\sqrt{2\pi}} e^{\frac{-(x-\mu)^2}{2\sigma^2}}$ $d = 2$; $\vartheta = (\mu, \sigma)$; $\Theta = (\mathbb{R} \times \mathbb{R}^+)$

Statistic T

A statistic is a *known* function of the data X. Usually, the dimension of T is much smaller than the dimension of X. Thus, a statistic simplifies the original data. A statistic is a function of the data/the draw and therefore a random variable.

Closely linked to the concept of a statistic is the statistical functional, which is a function(al) of the distribution $f(\cdot)$. In many cases, the distinction is not made and both are called “statistic”.

7.4. The Monte Carlo method

The Monte Carlo method is a numerical method of solving mathematical problems by random sampling. Simpler put, it is a method for calculating expectations. It was developed at the Los Alamos National Laboratory, the first US atomic bomb factory to calculate neutron diffusion in atomic bombs. The first paper about this method by Metropolis and Ulam dates from 1949. The Monte Carlo method got its name after the city of Monte Carlo, which is famous for its casinos, see Fig. 7.4. Note that the spinning wheel in a casino is one of the simplest mechanical devices to produce random numbers.

Example 14. The number of points on one side of a dice be x. What is the expected value $E[x]$ if we throw a dice?

Theoretical calculation. $E[x] = \frac{1}{6}(1+2+3+4+5+6) = 3.5$

Experiment. Roll a real dice 20 times, take the average. Result: $\mu(x) = \frac{69}{20} = 3.45$.

MATLAB simulation. Let MATLAB roll the dice n times and calculate the mean.

Example: A first simple simulation program `dicer.m`

```
n=100;                    % Good style: define n in the beginning
dice=ceil(6*rand(1,n));   % Roll 100 dice in one line
disp(dice(1:20));         % output the first few events for verification
disp(mean(dice));
```

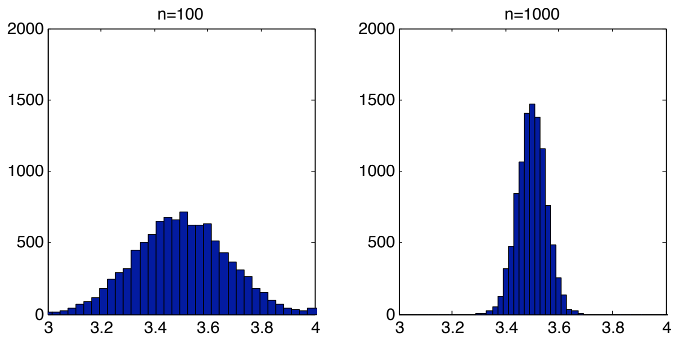


Figure 7.1.: The central limit theorem in action: Increasing `n` yields a more precise simulation result. Each simulation was repeated 10'000 times using `manydice.m`

Observations

- The results are somewhat close to the theoretical value, but not exact.
- Every time we repeat the simulation, we get a slightly different value.
- If we repeat the simulation many times, we can plot a distribution, see Fig. 7.1.
- If we repeat the simulation with `n=1000`, the distribution becomes more narrow.

Figure 7.2.: The casino of Monte Carlo, which gave the Monte Carlo method its name.

Conclusions

- The mean is a good estimator for the expected value.
- The result of a simulation is a random variable.
- By repeating the simulations, we can assess its properties.
- If we increase `n`, the variance of this random variable decreases.

Two questions immediately arise. (1) Can we always use the mean as an estimator for the expectation or have we just been lucky? (2) Does the variance of the estimation always decrease if we increase `n`? By how much? The next sections will answer them.

Why simulate? Example 14 is not breathtaking. We got a result (≈ 3.5) that we already knew, just with less precision. So is simulation any good? Not, if we are able to calculate the theoretical values. In many cases, this is, however, not possible or not practical.

Simulation is the method of choice for stochastic problems that are either too difficult to solve analytically or for which it is known that analytical solutions do not exist. The only thing we usually need for a simulation is a model and the problem definition.

There are two drawbacks: first, we obtain only a specific solution for specific parameters. Simulation methods cannot do anything about this, but we are free to run siumlations for many different parameter sets. The second disadvantage – that our result is obtained in the form of a random variable – can be easily mitigated by the fact that we can (a) assess the precision of our simulation result and (b) improve it, if necessary.

7.4.1. The method to calculate x

To calculate an unknown quantity x, we have to first construct a *probabilistic model* (often quite obvious) and follow a few simple steps.

1. Find a random variable ξ such that $E[\xi] = x$.
   Very often, this is trivial, as the desired quantity is $E[\xi]$.
2. Produce n independent random draws $\xi_1, \xi_2, ..., \xi_n$.
3. Use mean as estimator for the expected value (thanks to the LLN): $\hat{x} = \hat{\mu}_n = \frac{1}{n}\sum \xi_i$.
4. Estimate $Var(\mu)$ (see below, MCSE) and verify if it is below the desired value. If not, increase n accordingly and repeat steps 2 and 3.

Reflection. At first sight, step (1) may seem difficult. How could one find a suitable ξ that fulfills the condition $E[\xi] = x$ in order to estimate x? Fortunately, in many economics problems, the unknown quantity *is* the expectation $E[\xi]$. In this case, the problem is already defined in the sense of step 1.

Example 15 (Risk-neutral asset pricing). A risk-neutral agent values any asset as the discounted expected payoff.

$$p_0 = e^{-rt} E_t^Q[\xi_t] \tag{7.6}$$

With p_0 the asset price today (at time 0), r the risk-free interest rate, t the time until the payoff is paid and ξ_t the payoff at time t. Notice that ξ_t is a random variable (we assume we cannot predict asset returns). This problem is already formulated as in step (1) above.

7.4.2. Law of Large Numbers – LLN (Kolmogorov)

Given a sequence of n independent, identically distributed (i.i.d.) random variables ξ_i with

$$E[\xi_i] = \mu \tag{7.7}$$

Define the running sum S_n and the running average X_n as

$$\begin{aligned} S_n &= \sum_{i=1}^{n} \xi_i \\ X_n &= \frac{1}{n} S_n = \frac{1}{n} \sum_{i=1}^{n} \xi_i \end{aligned} \tag{7.8}$$

Then

$$X_n \xrightarrow{\text{a.s.}} \mu \tag{7.9}$$

The law of large numbers is *the* basis for simulation-based methods like the Monte Carlo method. It ensures that our simulation actually converges to the desired value. If we took an infinite number of draws, we would get the precise value of the expectation.

7.4.3. Central limit theorem – CLT (Lindberg-Levy)

Given a sequence of n i.i.d. random variables ξ_i with $E[\xi_i] = \mu, Var[\xi_i] = \sigma^2$. We first define a new random variable Y_n

$$Y_n = \frac{S_n - n\mu}{\sigma\sqrt{n}}.$$

The CLT states that Y_n converges in distribution to a standard normal distribution:

$$Y_n \overset{\text{distr}}{\longrightarrow} N(0,1)$$

An alternative formulation is:

$$X_n \overset{\text{distr}}{\longrightarrow} N(\mu, \frac{\sigma^2}{n}) \tag{7.10}$$

The CLT proves our observation that increasing `n` lowers the variance of the estimate. Furthermore, we can quantify this effect: if we increase `n` by a factor of 100, then the variance of the simulation will decrease by a factor of 100; the standard deviation will decrease by a factor of 10.

The advantage of simulation based methods such as the Monte Carlo method is that by increasing n, the variance can be made arbitrarily small. The drawback is the slow convergence speed and the computational intensity: to achieve 10 times the precision, we need 100 times the number of events.

The variance $\frac{\sigma^2}{n}$ is a measure for the precision of the Monte Carlo method. In theory, it could be made arbitrarily small.

Note: The σ is the variance of the *whole process*. It is not always possible to calculate σ. This σ is completely different from a possible σ-statistic *in the process*.

7.4.4. Monte Carlo Standard Error (MCSE)

The CLT gives the precision of the Monte Carlo method in stating that $X_n \overset{\text{distr}}{\longrightarrow} N(\mu, \frac{\sigma^2}{n})$. By increasing n, the variance decreases. But what is the value of σ^2?

Example 14 continued. In some cases, it is possible to calculate σ^2. In our example, ξ has a discrete distribution with equally probable values of 1 to 6. Then $V(\xi) = E\Big[\big(\xi - E[\xi]\big)^2\Big] = \frac{1}{6}[(-2.5)^2 + (-1.5)^2 + (-0.5)^2 + 0.5^2 + 1.5^2 + 2.5^2] \approx 2.92$. For $n = 1000$, the variance of a Monte Carlo estimate of $E[\xi]$ should be $2.92/1000 = 0.0029$. We verify this by running the program `manydice.m` with $n = 1000$ and calculating `var(result)`. In a test run, we obtain a value of 0.0028.

In the more interesting cases, in which already $E[\xi]$ is not accessible, we cannot calculate $V[\xi] = \sigma^2$, so we have to estimate it. It would be desirable to find an estimator $\hat{\sigma}$ that does not require to repeat the whole simulation several times. This estimator is the Monte Carlo Standard Error:

$$\hat{\sigma}_n = \sqrt{\frac{1}{n}\sum_{i=1}^{n}(\xi_i - \hat{\mu}_n)^2} \tag{7.11}$$

with $\hat{\mu}_n = \frac{1}{n}\sum \xi_i$ being the Monte Carlo estimate. The index in $\hat{\sigma}_n$ indicates that the standard error has been estimated using a sample size of n.

The MCSE is called "standard error", because it is standardized to a sample size of one – if we ran a Monte Carlo with $n = 1$, then the standard deviation of the result would be $\hat{\sigma}$. For larger samples, we apply (7.10). We get the MCSE $\hat{\sigma}_n$ with only one additional calculation after the Monte Carlo simulation. In most cases, producing the ξ_i is computationally quite intensive, so $\hat{\sigma}$ comes almost "for free".

Application. The MCSE can be used to determine the sample size that is required for some target precision. A common approach is to perform a preliminary simulation with $n = 1000$ and to calculate $\hat{\sigma}_n$. If $\hat{\sigma}_n$ is already lower than the desired precision, we are done. If not, we can calculate the number of draws N^* to achieve a target precision σ_t as

$$N^* = \frac{\hat{\sigma}_n^2}{\sigma_t^2} \tag{7.12}$$

7.4.5. Convergence, Monte Carlo versus lattice methods

Why is random sampling superior to a simple lattice/grid? The problem with a grid is that it involves a lot of points, especially in high dimensions. A lattice of N draws in d dimensions requires N^d calculations. To achieve a similar result with the Monte Carlo method, only $N \times d$ calculations are required – a lot less! More precisely, the relative errors ($\hat{\sigma}$) of the two approaches scale totally differently:

$$\begin{aligned} \epsilon_{lattice}(N,d) &\propto N^{-\frac{2}{d}} \\ \epsilon_{MC}(N,d) &\propto N^{-\frac{1}{2}} \end{aligned} \tag{7.13}$$

7.4.6. Extensions

General functions. The Monte Carlo method is not limited to estimating $E[\xi]$, it can in fact estimate $E[g(\xi)]$ for almost any well-behaved $g(\cdot)$. Furthermore, we can calculate any statistic of ξ, not only the expected value.

Variance reduction. A great deal of research has been devoted to making the Monte Carlo method faster and thus more efficient. All these techniques aim at reducing σ^2, which in turn makes less steps necessary to obtain a certain precision. The most popular techniques are **importance sampling**, in which the sample is drawn only out of the relevant part of the distribution of ξ and **antithetic sampling**, in which pairwise negatively correlated events reduce the overall variance. See P. Jaeckl: Monte Carlo Methods in Finance, Wiley 2002, chapter 10.

Quasi-random numbers

Can we beat the convergence speed of the central limit theorem? Yes, with quasi-random numbers: e.g. the Sobol sequence and the Halton sequence.

> MATLAB note useful commands for creating quasi-random numbers are `sobolset`, `haltonset` and the general command `qrandstream`.

7.4.7. Applications

Simulation and estimation – properties of an estimator

A large class of economic problems (e.g. asset pricing in finance) aim at explaining observed data with a probabilistic model. In such a model, the random variable ξ depends on a few parameters $a, b, \ldots$. In this setting, we want the result of the Monte Carlo simulation to be as close as possible to the data. The procedure is the following: (1) choose some parameters $a, b, \ldots$. (2) calculate the result of the MC. (3) compare it to the data. (4) if the result is close enough to the data, we are done. If not, we try some other parameters and continue with (2). In such a setting, we usually perform hundreds or thousands of MC simulation to estimate the optimal parameter set.

Properties of many estimators (consistency, efficiency, small-sample properties) can only be calculated using some assumptions (e.g. normality, homoskedasticity, no measurement error on y). In the real world, these assumptions are often violated. What happens, if the data is noisy, non-normal, heteroskedastic? This answer is often difficult to asses analytically $\rightarrow$ Do a Monte Carlo!

7.4.8. Four caveats

Simulation and differentiation/optimization

In these cases, we need to keep the seed the same.

Monte Carlo simulation and memory usage

Monte Carlo simulations may involve a lager number of tries, e.g. 1 million. Depending on the application, saving all the interim values may not be necessary. This can greatly reduce memory usage. See page 155.

Simulation and parallel computing

As we have seen, MATLAB produces exactly the same sequence of random numbers every time it is started. This is of relevance when using the parallel computing toolbox, as MATLAB simply launches identical copies of itself to perform parallel calculations. Each of these copies is newly started up and would produce exactly the same sequence of random numbers. If we want to split, say 1000 Monte Carlo simulations along 10 computers, we have to take care that each computer has a different seed when producing the random numbers for the simulation. For more details, see Chapter 14.

Simulate processes or distributions?

Whenever possible, we will try to avoid simulating the whole process, as this is considerably more intensive on the computational side. It makes no sense to simulate gaussian paths of a Black-Scholes model, when the distribution at time T is known analytically. Two circumstances make it necessary to simulate full paths:

- If the path is explicitly needed, e.g. for path dependent (asian) options
- If the distribution of the random variable X at $T = t + \tau$ is not known

7.5. PC-Lab: Simulation and estimation

`diary('simul.txt')`	Don't forget to start a diary.

First steps

`ceil(6*rand(1,3))`	Throw three dices.
`ceil(6*rand(10,3))`	Throw three dices ten times.
`sum(ceil(6*rand(10,3)),2)`	The sums of points of three dices; ten times.
`a=sum(ceil(6*rand(1000,3)),2);`	The sums of 1000 throws of three dices.
`hist(a,15)`	Histogram.

OLS properties

Linear model $Y = \beta X + \varepsilon$. (For more details, see Appendix C.) We run an OLS with simulated data:

1. We choose the dimension of the model. K=3, N=200.
2. We define $\beta = (1, 2, 5, 0.5)'$. (β has dimension $K + 1$).
3. We produce random X-values and a random error term.
4. We calculate Y from these.
5. Next, we estimate $\hat{\beta}$ from our results and compare it to the initial β.

`K=3, N=200`	Parameters.
`b=[1 2 5 0.5]'`	Chosen ("real") β. Slope parameter is 1.
`X=[ones(N,1) rand(N,K)];`	Include row of ones in X.
`e=randn(N,1);`	Produce the error term.
`Y=X*b+e;`	This is the linear model, Eq. 5.3.
`[bHat, binv]=regress(Y,X)`	Now we estimate $\hat{\beta}$.
`[b bHat]`	Compare the results.

The OLS estimate is a random variable

`e=randn(N,1);`	We produce a new error term ...
`Y=X*b+e;`	... and accordingly a new Y
`[bHat2, binv]=regress(Y,X)`	We perform the regression a second time.
`[b bHat bHat2]`	The results are slightly different now.

Violating OLS assumptions

Violate MLR.2: Zero conditional mean: $E(\varepsilon|X) = 0$

`e=0.5*randn(N,1)+0.5*X(:,2);`	Produce a correlated error term.
`Y=X*b+e;`	Produce a new Y with the new error term.
`[bHat, binv]=regress(Y,X);`	We now perform a new regression.
`[bHat b]`	Compare $\hat{\beta}$ to the true β. The second component – the one which is correlated to the error term – is biased.

Violate MLR.4: No perfect colinearity: $rk(X) = k + 1$

`X=[ones(N,2) rand(N,K-1)];`	MLR.4 is violated.
`rank(X)`	Verify that MLR.4 is violated.
`X(1:10,:)`	Verify what we produced.
`e=randn(N,1);`	Produce an unbiased error term.
`Y=X*b+e;`	Produce a new Y with the "bad" X.
`[bHat, binv]=regress(Y,X);`	Enjoy the error message!

Time series of stock prices

A very simple assumption about the stock market: prices vary randomly, going up and down with equal probability.

$$\frac{x_{t+1} - x_t}{x_t} = \frac{\Delta x_t}{x_t} = \xi \qquad \text{with } \xi \sim N(0, \sigma^2)$$

Let us simulate a time series of such a return.

`s=1`	Set $\sigma = 1$.
`r=s*randn(100,1)`	Interpret as a time series of 100 returns.
`plot(r)`	This looks pretty chaotic.
`p=100+cumsum(r)`	This is the running sum of the stock returns. If you are not sure what this line does, hit `help cumsum`.
`plot(p)`	This plot already looks a lot like a stock chart. (It is indeed the chart of $\log(x_t)$).

Exercise: Repeat this section several times and compare the plots. We created various scenarios of a stock price process!

`r=s*randn(100,30)`	Now we produce 30 scenarios at once.
`p=100+cumsum(r)`	Stock returns.
`plot(p)`	Plot of 30 scenarios.

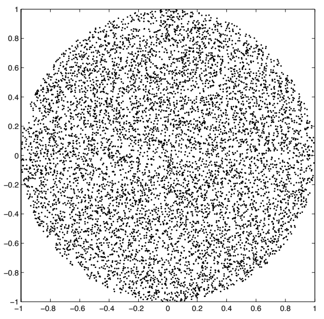
Question: How much would a risk-neutral investor pay for such an asset? She would pay the expected payoff = the expected value of the price on the last day. (We ignore interest and discounting). We can estimate this. It is the mean of p on the last day.

`mean(p(end,:))`	Of course, this is a rough estimate. (Why?)
`diary off`	Don't forget to mail the diary to yourself.

Calculating π

We first need a probabilistic model of π. The idea is to draw random points out of the unit square. If $x^2 + y^2 \leq 1$, then they are part of the unit circle, else they are not. π is the fraction of points that are in the circle times 4.

The program `circle_mc.m` produces the following figure:



Option pricing using the Monte Carlo method

Run the file `option_sobol.m` and read the comments.

7.6. Exercises

Exercise 7.1. Simulate the following stochastic processes. Use T=100, $x_0 = 1$ and sensible values for all other parameters. Do not use any toolbox functions. Perform a `clear` before every sub-point (a), (b), ...

(a) AR(1): $x_t = c + \varphi x_{t-1} + \varepsilon_t, \quad \varepsilon_t \sim N(0, \sigma^2)$

(b) AR(2): $x_t = c + \varphi_1 x_{t-1} + \varphi_2 x_{t-2} + \varepsilon_t, \quad \varepsilon_t \sim N(0, \sigma^2)$

(c) The general AR(p) process: $x_t = c + \sum_{i=1}^{p} \varphi_i x_{t-i} + \varepsilon_t, \quad \varepsilon_t \sim N(0, \sigma^2)$. Assume a vector φ of length p is given.

(d) MA(1): $x_t = \varepsilon_t + \theta_1 \varepsilon_{t-1}, \quad \varepsilon_t \sim N(0, \sigma^2)$

(e) The ARMA(p,q) process: $x_t = c + \varepsilon_t + \sum_{i=1}^{p} \varphi_i x_{t-i} + \sum_{k=1}^{q} \theta_k \varepsilon_{t-k}, \quad \varepsilon_t \sim N(0, \sigma^2)$. Assume a vectors φ and θ of correct lengths are given.

(f^*) The Feller diffusion process, see example 12 on page 71.

Exercise 7.2. Monte Carlo basics.

(a) Produce a vector `r1` with N=1000 uniformly distributed random numbers on $[-0.5, +0.5]$. Produce a vector `r2` with N=1000 random numbers that are normally distributed with mean zero and variance 0.5. Produce a vector `r3` that is an element-by-element sum of `r1` and `r2`.

(b) Calculate the sample mean and sample variance of `r3`.

(c) Calculate analytically the mean and variance of the underlying distribution of `r3`. Write the result as a comment in your program. *Hint:* `r1` and `r2` are uncorrelated.

(*d*) Re-do steps (a) and (b). Do you obtain the same result? Write a short explanation in a comment.

(*e*) Apply (7.11) to estimate $\hat{\sigma}$. Apply (7.12) to find N^* with $\sigma_t = 0.03$.

(*f*) Run 100 Monte Carlo simulations of `r3` with $N = N^*$ from question (*e*). Calcualte the fraction of runs in which the mean of `r3` is within ± 0.03 of the true value. Compare this fraction to the theoretical value in a short comment.

(*g*) Continuing with the same 100 Monte Carlo simulations from (f), calcualte the fraction of runs in which the variance of `r3` is within ± 0.03 of the true value. Compare this fraction to the theoretical value in a short comment.

(*h*) Repeat the whole exercise with `r1` $\sim U(-1, 1)$ and `r2` $\sim N(0, 5)$.

Exercise 7.3. Numerically find the probability for scoring exactly N points with M throws of a regular (six-sided) dice.(*a*) Start by producing all possible combinations and count the winning combinations. (*b*) Solve the problem using a Monte Carlo simulation.

Exercise 7.4. Find the probability for the case that a) exactly two persons b) at least two persons in a room have the same birthday (day+month). Assume that all years have 365 days and that birth dates are evenly distributed among the population. Calculate the above probabilities for every group size from 2 to N. Make one plot with both probabilities as a function of group size. Solve this problem using the Monte Carlo method.

Exercise 7.5 (Needle problem)**.** Imagine you have a piece of ruled paper with lines that are d centimeters apart. Now you drop a needle that is $l \leq d$ centimeters long on this piece of paper. What is the probability that the needle will cross one of the lines? Solve this problem via Monte Carlo simulation. (*a*) Calculate the probability for the special case $d = 1$ and $l = 0.5$. (*b*) Produce a plot of the probability as a function of $l \in [0, 1]$ for $d = 1$. (*c*) Produce the same plot as in (*b*), but for $l \in [0, 2]$ and $d = 2$. Can you find a general rule?

Exercise 7.6. (*a*) Simulate the double normal distribution from exercise 3.6 using $\mu_1 = 0, \mu_2 = 1, \sigma_1 = 1, \sigma_2 = 0.1$ and calculate the skewness and kurtosis. (*b**) Simulate several double normal distributions with $0 \leq \mu_i \leq 1$ and $0.1 \leq \sigma_i \leq 10$. Calculate for each of these distributes the skewness and kurtosis. Make a plot of skewness versus kurtosis of all your simulations. Can you find some kind of relationship between the two?

Exercise 7.7 (Up to you.)**.** Find 3 concrete applications of simulations in your field. (Examples and exercises in this chapter do not count).

8. Algorithms

References: Mathews and Fink, section 2; Brandimarte section 2.4; Judd, section 5; Moler, Kharab and Guenther (general and detailed), (Stanoyevitch 2005) sec 6

8.1. Nonlinear functions

8.1.1. Standard numerical problems with nonlinear functions

Standard problems in numerical mathematics for nonlinear functions $f(x)$:

- Roots: find (all) x such that $f(x) = 0$
- Fixed points: find (all) x such that $f(x) = x$
- Inverse function: find x such that $f(x) = y$ for a given y.
- Approximation: find a simple function $g(\cdot)$ such that $g(x) \approx f(x)$ near x_0
- Minimum: find x such that $f(x)$ is a global/local minimum ($\rightarrow$ next section).

Closely linked to the approximation problem is a problem in dealing with data:

- Interpolation: find $f(x)$ with $x_0 < x < x_1$, if only $f(x_0)$ and $f(x_1)$ are known

8.1.2. Properties of functions

There is no "one size fits all" numerical algorithm. Almost all methods presented here work by assuming a certain property of the function $f(\cdot)$. If this condition is not fulfilled, the numerical method may not work, or worse, it may work, but produce incorrect results. The use of a inappropriate numerical methods is a widespread source of errors.

Before using a numerical method, verify whether all conditions are fulfilled.

Definition 1 (Global vs. local). All of the following properties may apply *globally*, i.e. over the whole real line ($\forall x \in \mathbb{R}$); or *locally*, i.e. within an interval $x \in [x_{min}, x_{max}]$ or within a neighborhood $|x - x_0| < \delta$.

In many real world applications, there is no proof that a function has global properties. However, it can often be shown that the desired properties are present locally. Thus we

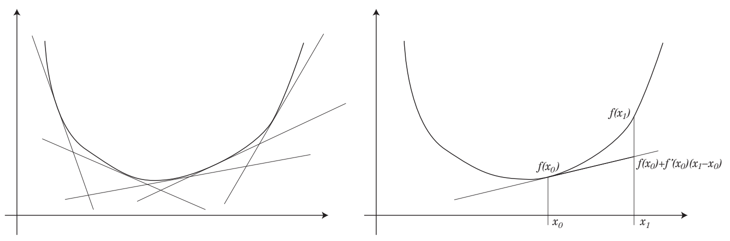


Figure 8.1.: Left: A convex function is always above its tangent.
Right: a convex function is always above its left-sided approximation.

can still use most numerical methods, provided we limit ourselves to the interval in which the function has the required property.

Definition 2 (Increasing/decreasing). An increasing function fulfills the inequality

$$f(x) \leq f(x+\Delta) \qquad \text{for } \Delta > 0 \tag{8.1}$$

An alternative definition is $f'(x) \geq 0$.

For a decreasing function, the inequalities are $f(x) \geq f(x+\Delta)$ and $f'(x) \leq 0$. A function is called strictly increasing/decreasing, if the strict inequality holds.

Definition 3 (Monotonicity). A function that is either increasing or decreasing is called monotonous. An alternative test for monotonicity is the horizontal line test: the graph of the function must cross any horizontal line at most once. Strictly monotonous functions are invertible and vice versa.

Definition 4 (Continuity). The classical definition for a continuous function is that for every $\epsilon > 0$ there exists a δ such that

$$f(x) + \epsilon = f(x+\delta) \tag{8.2}$$

An alternative description is that the graph of a continuous function can be drawn in one strike, without lifting the pen from the paper.

Definition 5 (Convexity/concavity). A convex function fulfills the inequality

$$\lambda f(x_1) + (1-\lambda) f(x_2) \geq f(\lambda x_1 + (1-\lambda) x_2) \tag{8.3}$$

Alternatively, a convex function is always above its tangent (see Fig. 8.1) or $f'' > 0$. In the multivariate case this corresponds to the condition that the Hessian matrix $H = \frac{\partial^2 f}{\partial x_i \partial x_j}$ is positive definite. An convex function is always above its left-sided approximation:

$$f(x_1) - f(x_0) > f'(x_0)(x_1 - x_0) \qquad \text{for } x_1 > x_0$$

For a concave function, the relation in (8.3) is inverted. It is always below its tangent, the Hessian matrix is negative definite and $f'' < 0$. A globally convex (concave) function has only one minimum (maximum). If $f(\cdot)$ is convex, then $-f(\cdot)$ is concave and vice versa.

Examples. The function $f(x) = e^x$ is strictly increasing and convex; $f(x) = \ln(x)$ is strictly increasing and concave; $f(x) = x^2$ is convex, decreasing for $x < 0$ and increasing for $x > 0$. See `fn_properties.m`

8.2. Polynomials, approximation and interpolation

There are good reasons why polynomials are widely used in mathematics. The are easy to analyze and fast to evaluate, they are always continuous, differentiable and integrable, they have simple analytical solutions to most problems (e.g. integration, differentiation) and we have efficient methods for fitting them and finding their roots. Polynomials of order > 1 are nonlinear functions themselves.

MATLAB commands for polynomials

`a=[1 2 3];`	Start with a vector
`p=poly(a)`	Produce polynomial $(x-1)(x-2)(x-3)$
`p`	Coefficients of x^3, x^2, x and 1 (see note below)
`polyval(p,0)`	Evaluate above polynomial at $x = 0$.
`polyval(p,1)`	If we evaluate `p` at one of its roots, the result is 0.
`x=0:0.1:4;`	The command `polyval()` is ...
`y=polyval(p,x);`	vector compatible. We can use this ...
`plot(x,y)`	for plots.
`r=[1 -2 1];`	This is the representation of $1 \cdot x^2 - 2 \cdot x + 1 \cdot 1$.
`roots(r);`	Now we solve the quadratic equation $x^2 - 2x + 1 = 0$.
`roots(p)`	If we solve for the original polynomial, we receive again `a`.

NOTE: MATLAB's notation for polynomials `p=[p1 p2 p3 p4]` means $p_1x^3+p_2x^2+p_3x+p_4$. This is exactly the other way round as in most parts of the algebraic literature, where the notation often starts with the constant term $a_0 + a_1x + a_2x^2 + a_3x^3$.

Horner's method is an efficient way to evaluate a polynomial and an example of a useful algorithm. You can verify that MATLAB uses this method, as well by typing `edit polyval`. This will open the code behind the `polyval()` command. The idea is successive multiplication:

$$a_3x^3 + a_2x^2 + a_1x + a_0 = ((a_3x + a_2)x + a_1)x + a_0$$

8.2.1. Approximation

To approximate a function $f(x)$ means to replace it by a similar function $g(x)$ such that $f(x) \approx g(x)$ in an interval around x_0. Approxmation presents us with a trade-off: $g(x)$ is usually easier to compute and analyze than $f(x)$, but less precise. Thus we will only use approximations if there is a good reason, mostly increased speed in general or the possibility to perform parts of a calculation analytically. The latter is especially interesting as it facilitates the analysis of a complex problem and helps to avoid some numerical calculations altogether (e.g. replacing a numerical integration). In addition to providing fast solutions, approximations are often used to assess their properties, (e.g. limits) by focusing on leading terms.

By the Stone-Weierstrass theorem, any continuous function on a closed interval $[a, b]$ can be approximated as closely as desired using polynomials. All these properties make polynomials such as the Taylor polynomial the obvious choice for approximations.

Error analysis. As every approximation involves some sort of error, it is useful to introduce two definitions of error.

Definition 6 (Absolute error). Given the exact value x and the approximate value $\tilde{x}$, the absolute error E_{abs} is:

$$E_{abs} = |x - \tilde{x}| \tag{8.4}$$

Definition 7 (Relative error). Given x and $\tilde{x}$, the relative error E_{rel} is

$$E_{rel} = \frac{E_{abs}}{|x|} \tag{8.5}$$

NOTE: It is important to use the true value x and not $\tilde{x}$ as denominator in (8.5), as an overestimated $\tilde{x}$ would otherwise make the relative error seem smaller than it is.

Taylor polynomials are widely used for approximating a function. Note that these polynomials are developed around the value x_0 and represent a good approximation only within some neighbourhood around it.

$$f(x) \approx f(x_0) + \sum_{k=1}^{N} \frac{1}{k!} (x - x_0)^k f^{(k)}(x_0) \tag{8.6}$$

The notation $f^{(k)}(x_0)$ means the k-th derivative of $f(\cdot)$, evaluated at x_0. Note that every polynomial of order p is its own Taylor polynomial of order p.

A Taylor polynomial of order 1 is called linear approximation. The coefficients of the Taylor polynomial are usually calculated by hand (or via the Symbolic Mathematics Tool-

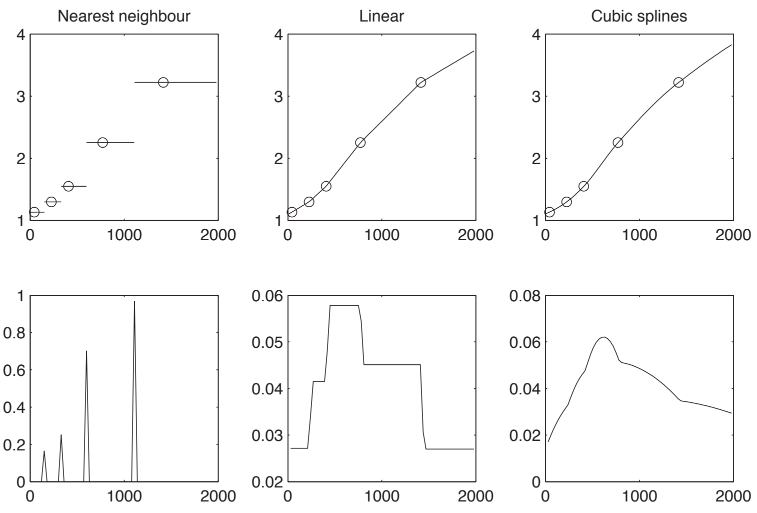


Figure 8.2.: Interpolation of the US yield curve on 2004-02-02. Duration in days. Top row: interpolated yields with observation points. Bottom row: first derivative.

box, see Sec. D.6). In the program we evaluate the polynomial.

8.2.2. Interpolation

Definition 8. The interpolation problem is to find the values of a function $f(x)$ for $x \in [x_1, x_n]$ if only $f(x_1), \dots f(x_N)$ are known. If x should also take values outside $[x_1, x_n]$, the operation is called *extrapolation*.

This definition has only one requirement, namely that the interpolation function $f(x)$ passes through all observed points (unlike for example a regression function).

Full polynomial interpolation. One naive way to interpolate between n observed points is a polynomial of degree (n-1). It is always possible to fit a polynomial of order $p-1$ through p points $\big(x_i, f(x_i)\big)$. In MATLAB this is done using the command `polyfit(x,y,n-1)`, where `x` and `y` are vectors of the observed points and `n` is the number of points. This solution will always exist and be continuous, but polynomial interpolation is plagued by [overshooting], for large n, see PC lab. Also, the assumption that the data generation function $f(x)$ is a polynomial of high order is normally not realistic. There are two alternatives to full polynomial interpolation.

Piecewise polynomial interpolation Piecewise interpolation makes fewer assumptions and usually produces fewer artifacts. It uses groups of n neighbouring points to fit a polynomial of order $n-1$, usually with $n \leq 3$. The simplest case is $n = 1$, i.e. a polynomial of order zero or a constant. This interpolation is called "nearest neighbor" interpolation.

As can be seen in the first column of Fig. 8.2, nearest neighbor interpolation is very close to the original data, as it only permits observed y-values, but it is not continuous.

The next case is $n = 2$, i.e. a polynomial of order 1 or linear interpolation. The equation for the linear interpolation/extrapolation is

$$f(x) = f(x_0) + \frac{f(x_1) - f(x_0)}{x_1 - x_0}(x - x_0) \tag{8.7}$$

The second column of Fig. 8.2 reveals that the interpolation function is now continuous, but its first derivative is not. This is due to the piecewise nature of the interpolation and would be the same for higher-order piecewise polynomial interpolations.

Cubic spline interpolation A spline is a piecewise defined function that is smooth, i.e. it is at least continuously differentiable. For each pair of points, a polynomial of order 3 is defined by the two points and a smooth pasting condition, i.e. that the first derivative be continuous. The detailed derivation is a bit tedious, but MATLAB provides a ready function for spline interpolation.

MATLAB commands for interpolation

```
yi = interp1(x,y,xi,method)
```

`x,y`	observed data points
`xi`	x-values of interest
`yi`	resulting interpolated values
`mode`	can be 'nearest', 'linear' or 'spline'

Transformation before interpolation

When performing an interpolation, we want to concentrate in the underlying phenomenon and reduce any predictable interpolation error. Therefore, it is a good idea to remove all exogenous variation before performing an interpolation. This is obtained by transforming the data before interpolating and applying the inverse transformation on the interpolation results. Formally:

1. Transform data: $\xi_{0,1} = T(x_{0,1})$
2. Interpolate $\xi_{0,1}$ to obtain ξ_2
3. Transform back: $x_2 = T^{-1}(\xi_2)$

Good transformations include transforming bond prices to interest rates (removing a duration-dependence of the interpolation error) of transforming option prices to implied volatilities.

Caveats and problems

Interpolation is always an interpretation of the data and can therefore produce misleading results. Interpolation may alter important statistical properties of a sample like its variance or autocorrelation. It is therefore especially dangerous to use interpolated data in any statistical estimation, see PC lab for an important example.

8.3. Numerical algorithms

Definition 9. An algorithm is a finite list of well-defined instructions for solving a problem, e.g. calculating a function. Any algorithm starts with an initial state ("input"). If a solution to the problem given the input exists, the algorithm terminates at a final state ("output"). The process of applying an algorithm to a specific input is called a *computation.*

Every algorithm must have four important properties:

1. The description must be unique and finite.
2. Each step must be calculable.
3. Each step must require only a finite amount of memory.
4. The algorithm must stop after a finite number of steps.

The above definition is very broad and can be applied to everything from folding a paper plane to solving a differential equation. For this script, we limit ourselves to numerical algorithms that calculate the value of a function. At first, this seems to be overly restrictive, but in fact every numerical application in finance can be expressed as the evaluation of a function.

8.3.1. Classification of algorithms

How can we choose the appropriate algorithm given a specific problem and available resources? Algorithms can be classified according to the following criteria, which can guide their assessment.

Domain An algorithm's domain specifies under which circumstances it can be applied. This usually implies some restrictions on the inputs, e.g. positivity of a variable or one of the five function properties listed in section 8.1.2.

Iteration An iterative algorithm repeats several instructions until a stopping condition is met, refining the solution at every step. As there is no "until"-loop in MATLAB, we usually implement these repetitive instructions via a `while` loop and the negation of the stopping condition. Section 8.3.2 is entirely devoted to iterative approximations.

Recursion A recursive algorithm invokes itself repeatedly until a stopping condition is met. Recursive functions can be directly implemented in MATLAB, see section 8.5.

Exact or approximate Approximations are used whenever the exact solution is not available or is too slow or cumbersome to implement. Approximate algorithms usually provide at least an upper boundary for the approximation error and in many cases it is possible to trade precision for speed, i.e. any desired precision may be obtained if enough computing power is available, see section 8.3.2.

Serial or parallel The problem of parallelization is not computer-specific: a tedious (manual) computation performed by a single mathematician can benefit from more brain power only if the according algorithm is parallel. Parallelization is relevant since modern hardware has more and more calculation units. See chapter 14.

Deterministic or stochastic Given a certain input, a deterministic algorithm produces exactly the same output at every run, while the result for a stochastic algorithm will vary slightly. Stochastic algorithms produce by definition only approximate results, but the precision of the result can usually be traded for computing power. *Example:* The Monte Carlo method is a stochastic algorithm for calculating the expectation of a random variable.

8.3.2. Iterative approximations and convergence

A large class of interesting algorithms are iterative approximations. They start with a "guess" and reduce the approximation error at every iteration. This has two consequences: first, as the error shrinks at every iteration, it can become arbitrarily small (within the limits of the floating point representation). Second, the choice of starting value becomes theoretically irrelevant, as a bad choice simply implies a larger starting error. The stopping condition of such an algorithm is usually defined by a tolerance – the maximal acceptable error – that needs to be undercut.

```
Make initial guess;
while $error > tolerance$
    improve guess;
end
```

Figure 8.3.: Pseudo-code of an iterative approximation algorithm.

Mathematically, the loop can be seen as producing a series of approximations S_i that converges to the exact solution: $S_1, S_2, \dots S_n \to S$. Equivalently, the sequence of relative approximation errors[1] $\epsilon_i = \frac{S_i - S}{S}$ can be said to converge to zero: $\epsilon_1, \epsilon_2, \ldots \epsilon_n \to 0$. This means we can borrow the concept of convergence speeds from the analysis of series.

Definition 10 (Linear convergence). A sequence ϵ_i is said to have linear convergence to 0 if

$$\lim_{n\to\infty} \frac{\epsilon_{n+1}}{\epsilon_n} \to q \qquad \text{with } q \in (0,1).$$

Special cases: if $q = 0$ the sequence is said to have superlinear convergence,
if $q = 1$ the sequence is said to have sublinear convergence.

Definition 11 (Logarithmic convergence). A sequence ϵ_i that converges sublinearly (see above) is said to converge logarithmically to zero if

$$\frac{\epsilon_{n+2} - \epsilon_{n+1}}{\epsilon_{n+1} - \epsilon_n} \to 1.$$

Definition 12 (Polynomial convergence). A sequence ϵ_i that converges superlinearly (see above) is said to have polynomial convergence to zero of order p if

$$\frac{\epsilon_{n+1}}{\epsilon_n^p} \to q \qquad \text{with } q \in [0,1] \text{ and } p > 0.$$

The case $p = 1$ is called linear convergence with $\epsilon_{n+1} = q\epsilon_n$ and $\epsilon_n = q^n \epsilon_1$. The case $p = 2$ is called quadratic convergence with $\epsilon_{n+1} = q{\epsilon_n}^2$ and $\epsilon_n = q^n {\epsilon_1}^{2n}$.

NOTE: Polynomial convergence of a high order is fastest, logarithmic convergence is slowest.

Example 16 (Babylonian method for calculating square roots.). This is one of the the oldest recursive algorithms:

1. Select an arbitrary start value $x_0 > 0$. Rule of thumb: $x_0 = S/2$.
2. Let $x_n = \frac{1}{2}\left(x_{n-1} + \frac{S}{x_{n-1}}\right)$
3. An upper boundary for the absolute error its: $e_n = \left| x_n - \frac{S}{x_n} \right|$
4. Repeat steps 2-3 while e_n is larger than the desired tolerance.

Notes: The algorithm even works if $x_0 > S$. One possible interpretation of this algorithm is that we use the arithmetic mean to approximate the geometric mean. The the error estimate comes from the fact that the true $\sqrt{S}$ must always lie between x_n and $\frac{S}{X_n}$, thus the width of the interval is the upper boundary.

[1]Sometimes it is easier to show convergence of absolute errors $\epsilon_i = (S_i - S)$. In the limit, both statements are equivalent and convergence rates are of the same order.

Let us now show the convergence of this algorithm and find its convergence speed. Note that in the proof we use the true value $\sqrt{S}$ and the true error as opposed to the upper boundary e_n. First we show a simple identity:

$$\begin{aligned}
\epsilon_n &= \frac{x_n - \sqrt{S}}{\sqrt{S}} = \frac{x_n}{\sqrt{S}} - 1 \qquad \text{(Definition of the relative error)} \\
x_n &= \sqrt{S} \cdot (1 + \epsilon_n) \qquad (8.8)
\end{aligned}$$

Next we express ϵ_{n+1} in terms of ϵ_n:

$$\begin{aligned}
\epsilon_{n+1} &= \frac{x_{n+1}}{\sqrt{S}} - 1 = \frac{\frac{1}{2}\left(x_n + \frac{S}{x_n}\right)}{\sqrt{S}} - 1 = \frac{\sqrt{S} \cdot (1+\epsilon_n) + \frac{S}{\sqrt{S}\cdot(1+\epsilon_n)}}{2\sqrt{S}} - 1 \\
&= \frac{1}{2}\left(1 + \epsilon_n + \frac{1}{1+\epsilon_n}\right) - 1 \quad = \quad \frac{\epsilon_n^2}{2(1+\epsilon_n)} \qquad (8.9)
\end{aligned}$$

Now we can evaluate $\epsilon_{n+1}/\epsilon_n$:

$$\frac{\epsilon_{n+1}}{\epsilon_n} = \frac{\epsilon_n}{2(1+\epsilon_n)} < 1 \qquad (8.10)$$

How fast is ϵ_n approaching zero? To assess the convergence speed, we have to go back to (8.9) and distinguish two cases:

$$\begin{aligned}
&\text{Case 1: big errors} \qquad \epsilon_{n+1} \approx \frac{\epsilon_n}{2} \quad \to \quad \frac{\epsilon_{n+1}}{\epsilon_n^{\mathbf{1}}} \approx \frac{1}{2} \quad \text{linear convergence} \\
&\text{Case 2: small errors} \qquad \epsilon_{n+1} \approx \frac{\epsilon_n^2}{2} \quad \to \quad \frac{\epsilon_{n+1}}{\epsilon_n^{\mathbf{2}}} \approx \frac{1}{2} \quad \text{quadratic convergence}
\end{aligned}$$

8.4. Some examples for iterative algorithms

The two problems of root finding and function inversion are actually very similar: root finding can be seen as inverting a function with $y = f(x) = 0$, inverting a function can be seen as finding the root of a function $g(x) = f(x) - y$.

8.4.1. Newton's method for root finding

Newton's method is an algorithm to find the root of a nonlinear function $f(x)$, with known first derivative, and which is monotonous on an interval. The idea is illustrated in Fig. 8.4.

1. Choose a starting value x_0. This value must be in an interval around the root in

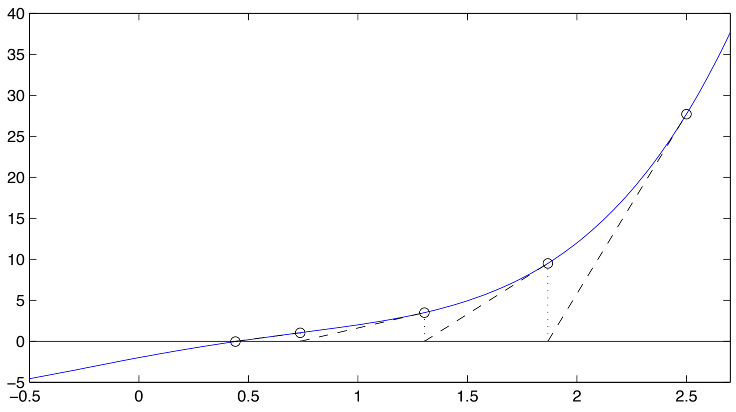


Figure 8.4.: An illustration of Newton's method, produced by `mynewton.m`.

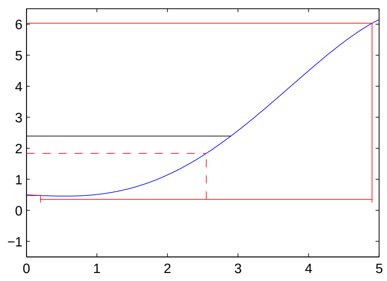
which $f(x)$ is monotonous.

2. Draw the tangent in (x_n, y_n), find the new value x_{n+1} where the tangent intersects with the x-axis.
3. Redo step 2 until y_n is below the precision threshold.

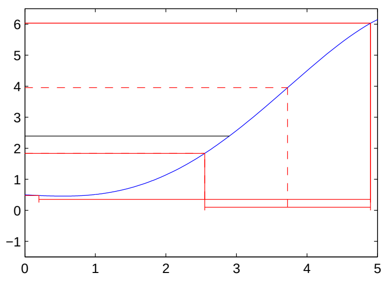
Note the fast convergence, which depends on the second derivative. If $f(x)$ were a line, the algorithm would hit the true value after one step. For moderately curved functions, the Newton method converges very fast.

8.4.2. Bisection for inverse functions

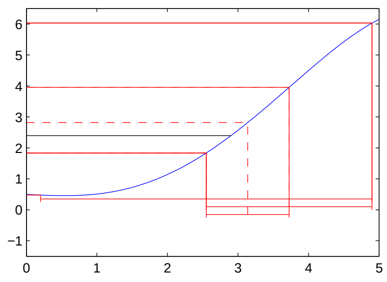
Bisection is an algorithm to invert a function, i.e. to calculate $x_0 = f^{-1}(y_0)$ given that $f()$ is strictly increasing within a chosen starting interval that includes x_0 (i.e. $f()$ is locally increasing). This monotonicity condition ensures invertability, as $f : X \to Y$ is a bijection: for every $y \in Y$ there is exactly one $x \in X$ such that $y = f(x)$.



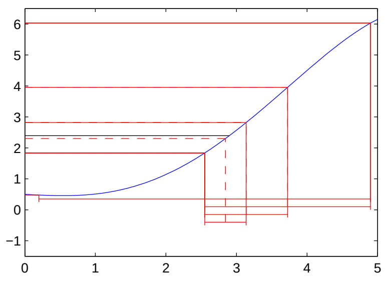
(1) Bracketed, choose right.



(2) Choose left.



(3) Choose left.



(4) Choose right.

Figure 8.5.: Illustration of the first four steps bisection method, produced by `mybisection.m`. The black line indicates y_0.

(1) Guess a starting interval $[x_L, x_R]$, calculate $y_L = f(x_L)$ and $y_R = f(x_R)$. These intervals are shown with red lines in Fig. 8.5.

(2) If $y_L < y_0 < y_R$ then y_0 is *bracketed.* If not, try a wider starting interval.

(3) Split the interval in the middle, set $x_M = \frac{1}{2}(x_L + x_R)$, and calculate $y_M = f(x_M)$. This split is shown as a dashed red line in Fig. 8.5.

(4) If y_0 is in the left interval i.e. $\min(y_L, y_M) < y_0 < \max(y_L, y_M)$[2], choose it as the new interval (i.e. $x_R = x_M$), otherwise choose the right one (i.e. $x_L = x_M$). The width of the x-interval has been halved.

(5) Repeat (3-4) until the upper estimate for the error, $e_i = |x_R - x_L|$ is sufficiently small.

Convergence. The interval $[x_R - x_L]$ shrinks by half at every iteration, thus $e_i \to 0$ and $\frac{e_{i+1}}{e_i} = \frac{1}{2}$, i.e. linear convergence. In practice, the final error after N iterations is $\varepsilon_N = \frac{(x_R - x_L)}{2^N}$. Many functions in probability (e.g. the c.d.f.) are increasing over the whole real line and bisection can be readily applied.

Example 17 (Implied volatility). The price of a European call option is given in the Black-

[2]We formulate the criterion this way to make bisection work with decreasing *and* increasing functions.

Scholes formula:

$$C = f(X, S_t, \sigma, t, r) \tag{8.11}$$

Very often, call prices are known and we want to construct a model to explain their behaviour. Furthermore, X, S_t, t, r are also known precisely. This leaves σ as the obvious choice for a free parameter. The question is: which σ would have a risk-neutral investor have used to price this option? The σ that fulfills this equality is called the implied volatility. The price of every vanilla option is strictly increasing in σ, so one obvious choice is bisection.

$$\sigma = f^{-1}(C, X, S_t, \sigma, t, r) \tag{8.12}$$

NOTE: How to produce a plot of the IV-surface is shown in D.2

Bisection for root finding

If we set $y = 0$, bisection can be used for root finding. This is actually done in the MATLAB function `fzero()`.

Example 18 (Finding the implicit yield). Let us slightly change example on page 44. Let the given quantities be the initial investment x, the annual rent y, the duration t and the present value PV_0. Which is the implicit interest rate r? It is the one for which $PV_0 = \sum \frac{CF_i}{(1+r)^i}$ or, better, $PV_0 - \sum \frac{CF_i}{(1+r)^i} = 0$. Thus the problem is reduced to the root of the difference.

We first write a function `PV` that calculates the present value for a given interest rate:

```
function result = MyPV(x, y, t, r)
% INPUT  x 1x1 .... initial payment
%        y 1x1 .... yearly payment
%        t 1x1 .... number of years
%        r 1x1 .... interest rate
result = -x;                        Initialize sum: value of year 0.
for k = 1: t                        We receive payments over 20 years
  result = result + y/(1+r)^k;      Running sum; every year we add to PV
end                                 (of function)
```

Next, we find the root of $PV - PV_0$:

```
% impyield.m Demo of fzero
% Peter Gruber, MATLAB course 2007
x = 1000000; y= 100000; t=20;
PVstar=242000;
diff=@(r)MyPV(x,y,t,r)-PVstar;
fzero(diff,0)
```

8.5. Recursive functions

A recursive function is a function that calls itself repeatedly until a stopping condition is met. This elegant mathematical concept can be directly translated to MATLAB, by simply calling a function within itself. However, it is not always optimal to implement recursive functions in MATLAB. Technically, MATLAB has to track every recursion in a list (the "stack") to correctly finish all the calls of the recursive function. To avoid this list from growing to long, MATLAB limits the number of recursions. The limit can be obtained using `get(0,'RecursionLimit')` and changed using `set(0,'RecursionLimit', N)`. Currently, the default limit is 500, which may be too small for many applications.

We advocate to avoid recursive functions in MATLAB. Luckily, there is always an iterative representation for a recursive algorithm and in most cases, it will not be more difficult to implement.

Example 19. The factorial $n! = 1 \cdot 2 \cdot n$ can be recursively expressed by the factorial of $(n-1)$, namely $n! = n \cdot (n-1)!$ with the definition $1! = 1$. In MATLAB, we can implement this as follows:

```
function factN = myfactorial(n)
% INPUT  n     1x1 .... initial number
% OUTPUT factN 1x1 .... factorial of n
if n > 1
  factN = n*myfactorial(n-1);
else
  factN = 1;
end
```

NOTE: This function will never reach MATLAB's recursion limit, because the factorial of 171 is already larger than `realmax = 1.7977+e308`, the largest number that MATLAB can handle.

8.6. PC lab: Traps in polynomial interpolation

Horner's method

`x=1.7;`	Evaluate polynomial at `x = 1.7`
`sum=p(1);`	Initialize algorithm. (Re-use polynomial `p()` from above.)
`for k=2:length(p);`	Loop over elements of the polynomial.
`  sum=p(k)+sum*x;`	Taking care of the special notation in MATLAB.
`end;`	
`disp(sum)`	

`p=[1/24 1/6 1/2 1 1];`	Taylor polynomial of the exponential function. MATLAB notation for $f(x) \approx 1+x+\frac{x^2}{2!}+\frac{x^3}{3!}++\frac{x^4}{4!}$
`x=-3:0.1:3;`	Some x-values
`plot(x,exp(x));hold on`	Plot original fn, `hold on` overlays approx.
`plot(x,polyval(p(4:end),x));`	The 1st order polynomial.
`plot(x,polyval(p(3:end),x));`	The 2nd order polynomial.
`plot(x,polyval(p(2:end),x));`	The 3rd order polynomial.
`plot(x,polyval(p,x));`	Full 4th order polynomial.

Polynomial interpolation

```
x=[1 2 5 6 10];
N=length(x);
y0=3*x;
plot(x,y0,'k--'); hold on
y=3*x+1*randn(1,N);
p = polyfit(x,y,N-1);
plot(x,y,'o');
plot( 1:0.1:10, polyval(p, 1:0.1:10) )
% move 1 point a bit
y(3)=y(3)+0.5;
plot(x(3),y(3),'ro');
p = polyfit(x,y,N-1);
plot( 1:0.1:10, polyval(p, 1:0.1:10),'--')
hold off
%% repeat this
% the phenomenon is called polynomial wiggle

% now compare to piecewise linear interpolation
y=3*x+1*randn(1,N);
ytrue=3*(1:0.1:10);
p = polyfit(x,y,N-1);
plot(x,y0,'k--'); hold on
plot(x,y,'o');
plot( 1:0.1:10, polyval(p, 1:0.1:10) )
y1=interp1(x,y,1:0.1:10,'linear');
plot( 1:0.1:10, y1, 'r--');
hold off
figure
subplot(2,1,1)
plot(1:0.1:10,abs( polyval(p, 1:0.1:10)-ytrue) ); hold on
plot(1:0.1:10,abs( y1-ytrue ),'r--');
title('errors')
subplot(2,1,2)
hist([abs( polyval(p, 1:0.1:10)-ytrue).' abs( y1-ytrue ).' ] )
```

Tick data interpolation

```
% interpolation_hf.m
% peter.gruber@usi.ch, 2013-02-01

% setup

% create a high-frequency data set, sampled every 15 seconds
% 1 trading day = 8 hours = 480 mins = 1921 data points
% for simplicity mu=0;
sigma = 0.20 / sqrt(252) / sqrt(1921);

npoints15 = 1921;                % number of points
r0=sigma*randn(npoints15,1);     % 15 sec returns
y15  = exp(cumsum(r0));          % tick price data
x15  = 15*(0:npoints15-1);       % tick times
plot(x15,y15)                    % plot
hold on

% calculate returns and annualized sigma
retn = log(y15(2:end))-log(y15(1:end-1));
sigmaHat15=sqrt(var(retn)*252*npoints15)

% now we interpolate to 5 second intervals
npoints5 = 1920*3+1;                    % number of poins
x5 = 5*(0:npoints5-1);                  % new tick points
y5 = interp1(x15,y15, x5, 'linear');    % interpolation
plot(x5,y5,'r--')                       % plot exactly on top of 15s ...
    interval
retn = log(y5(2:end))-log(y5(1:end-1));
sigmaHat5=sqrt(var(retn)*252*npoints5)

% now we interpolate to 1 second intervals
npoints1 = 1920*15+1;                   % number of poins
x1 = 1*(0:npoints5-1);                  % new tick points
y1 = interp1(x15,y15, x1, 'linear');    % interpolation
retn = log(y1(2:end))-log(y1(1:end-1));
sigmaHat1=sqrt(var(retn)*252*npoints1)

% would the same thing have happend with nearest neighbour interpolation?
y1NN = interp1(x15,y15, x1, 'nearest'); % nearst neighbour interpolation
retn = log(y1(2:end))-log(y1(1:end-1));
sigmaHat1NN=sqrt(var(retn)*252*npoints1)

% Extra question: and if we had eliminated points?
% try a 60s grid
npoints60 = 1920/4+1;                   % number of poins
x60 = 60*(0:npoints60-1);               % new tick points
y60 = interp1(x15,y15, x60, 'linear'); % this is called ...
    "interpolation", but
                                        % actually we eliminate points
plot(x60,y60,'k--')                     % plot is now different (need to ...
    zoom in)
retn = log(y60(2:end))-log(y60(1:end-1));
sigmaHat60=sqrt(var(retn)*252*npoints60)
```

8.7. Exercises

Exercise 8.1 (Interpolation)**.** Start from the code example `myInterpolation.m`. (*a*) Calculate month-by-month forward interest rates using the different interpolation schemes. The continuously compounded forward interest rate between horizons s and t, $s < t$ is $f(s,t) = \frac{r(s)s-r(t)t}{t-s}$, all times are measured in years. Make a plot of the different forward interest rate curves. (*b*) Add the linear interpolation scheme to the comparison. Calculate the month-by-month forward variance swap rate, which is defined as $VS(s,t) = sqrt\frac{VS^2(s)s-VS^2(t)t}{t-s}$. Make a plot of the different forward variance swap rate curves.

Exercise 8.2. (*a*) Write a function `sqrtB` that calculates the square root of any number to a precision of 10^{-10} using the Babylonian algorithm. The function should have only one input and one output argument. (This implies that you have to choose your own starting value). (*b*) Write a vector-compatible version of your function that calculates the square roots of several numbers using only one loop.

Exercise 8.3 (Inverse quantile.)**.** Write a MATLAB function `quantileInverse()`, which takes a dataset X and a fraction $0 \leq p \leq 1$ and returns the largest number q, for which a fraction p of the dataset is still smaller than q. Write one version of the function using only basic MATLAB commands and one based on MATLAB's `quantile()` function.

Exercise 8.4 (Kramer's conjecture.)**.** Take any number $n \in \mathbb{N}$. If it is even, divide it by 2. If it is odd, multiply it by 3 and add 1. Apply the same rules to the result and so on. The conjecture states that you will always end up at 1. Write a short program that verifies Kramer's conjecture for the numbers between 1 and 100, using ...

(*a*) an iterative function that verifies Kramer's conjecture for one given input number.

(*b*) a recursive function that verifies Kramer's conjecture for one given input number.

(*c*) Assuming that 0 is an even number, what would happen if we verified the conjecture for zero? Try it and write a one-line answer as a comment to your main program.

Exercise 8.5 (Implied Volatility)**.** Using the results from Exercise 3.5, (*a*) write a function that inverts the Black Scholes formula to calculate the implied volatility of a call or put option. Note that both call and put prices are strictly increasing functions of the volatility. (*b*) For which variables would it make sense to vectorize the function? Write a vectorized version of your function.

Exercise 8.6 (Factorial)**.** Write an iterative version of the function `myfactorial()` in Example 19.

9. Numeric differentiation and precision

References: Mathews and Fink, sec 6; Judd, sec 2.5, 7; Moler; Kharab and Guenther, Gilli, Maringer and Schumann, sec 2

9.1. Introduction and mathematical background

The derivative is defined as a limit:

$$f'(x) = \lim_{h \to 0} \frac{f(x+h) - f(x)}{h} \approx \frac{\Delta f(x)}{\Delta(x)} \tag{9.1}$$

The problem of numeric differentiation is to find a suitable replacement for the limit $h \to 0$. Setting h outright zero would lead to the undefined expression 0/0. Thus the limit must be replaced by some "very small number". Two questions arise: What is the optimal "small number" and is there any other room for improvements?

Example 20. We compare the numerical derivative for different values of h to the known analytical result for $f(x) = x^3$ with $f' = 3x^2$ and plot the absolute error in Fig. 9.1.

```
f=@(x)x^3; x0=2;                    Define function and x0
truefprime=3*x^2;                   Define true derivative
h=10.^-(0:18); x=2;                 Different values of h
f_prime=( f(x+h)-f(x) )./h          Vector-valued calc. of all num. derivatives
loglog(h,abs(f_prime-truefprime));  Log/log plot of error as function of h
```

9.2. The loss of precision theorem

Before we can continue the discussion of the optimal numerical differentiation, we need to define a few concepts in the field of precision.

Theorem 1 (Loss of precision)**.** *Consider a subtraction $x - y$ with $0 < y < x$ such that*

$$10^{-q} \leq 1 - \frac{y}{x} \leq 10^{-p} \tag{9.2}$$

Then at most q and at least p significant decimals are lost in the subtraction $x - y$.

An approximation for the number of digits lost is

$$q \approx -\log_{10}(1 - \frac{y}{x}) \tag{9.3}$$

The result is usually not integer.

A few remarks:

- No precision is lost in multiplication, addition and division.
- The reason why a loss of precision occurs with the subtraction is that this is the only operation in which two very large operands can produce a very small result. This small number is cut off in the floating point number representation.
- If $x = y$, an infinite number of significant digits is lost.
- If x and y have different signs, they are actually added and there is no loss of precision.

MATLAB example for the loss of precision theorem

`x=1.23457;`	Define two similar numbers ...
`y=1.23456;`	with 6 digits of precision.
`format long`	
`-log10(1-y/x)`	Prediction of the number of digits lost.
`x-y`	Result has only one digit of precision, five lost.
`x=2.23457;`	Define a very large number.
`-log10(1-y/x)`	Not even 1 digit will be lost.

NOTE: The same holds for the number of significant bits lost: $2^{-a} \leq 1 - \frac{y}{x} \leq 2^{-b}$. At most

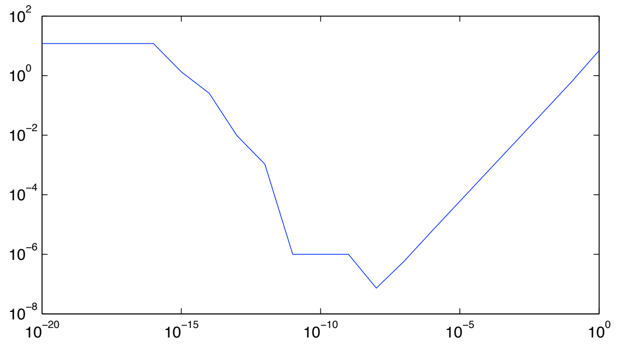


Figure 9.1.: Reducing h increases the precision only up to a point: there is an optimal h.

a and at least b significant bits are lost.

9.3. Tradeoff between numerical and analytical precision

How can we explain the shape of the error function in Fig. 9.1? There are two sources of error in numerical differentiation.

Truncation error. The first is the (analytical) approximation error for $h > 0$. We can use a Taylor expansion to estimate it:

$$f(x+h) = f(x) + f^{'}(x)h + \frac{f^{(2)}(x)}{2!}h^2 + \frac{f^{(3)}(x)}{3!}h^3 + \cdots \tag{9.4}$$

The error is the approximate derivative minus the true derivative:

$$\epsilon_A = \frac{f(x+h)-f(x)}{h} - f^{'}(x) = \frac{f^{(2)}(x)}{2!}h + \frac{f^{(3)}(x)}{3!}h^2 + O(h^3) \tag{9.5}$$

Numerical error. The second (numerical) error stems from the loss of precision theorem, because $f(x+h)$ gets very close to $f(x)$ for small h. To estimate the number of lost digits we note that the source of the precision loss is the operation $f(x+h)-f(x)$, independently of the subsequent division by h.

Assume $f(\cdot)$ is a positive increasing function, then $0 < y = f(x) < x = f(x+h)$ (if $f(\cdot)$ is decreasing, swap x and y). We rewrite (9.3) and insert the Taylor expansion (9.4).

$$\begin{aligned}
q &= -\log_{10}\left(1-\frac{y}{x}\right) = -\log_{10}\left(\frac{x-y}{x}\right) \\
&= -\log_{10}\left(\frac{f(x+h)-f(x)}{f(x)}\right) \\
&= -\log_{10}\left(\frac{f(x)+f^{'}(x)h - + \frac{f^{(2)}(x)}{2!}h^2 + \cdots - f(x)}{f(x)}\right) \\
&= -\log_{10}\left(\frac{f^{'}(x)h + \ldots}{f(x)}\right)
\end{aligned} \tag{9.6}$$

If q digits are lost, the corresponding absolute error 10^q times the smallest number than can be represented in the numerical system. In MATLAB this number is `eps`$= 2.2 \times 10^{-16}$.

$$\epsilon_N = \texttt{eps} \cdot 10^q = \texttt{eps} \cdot \frac{f(x)}{f^{'}(x)h} + O(1) \tag{9.7}$$

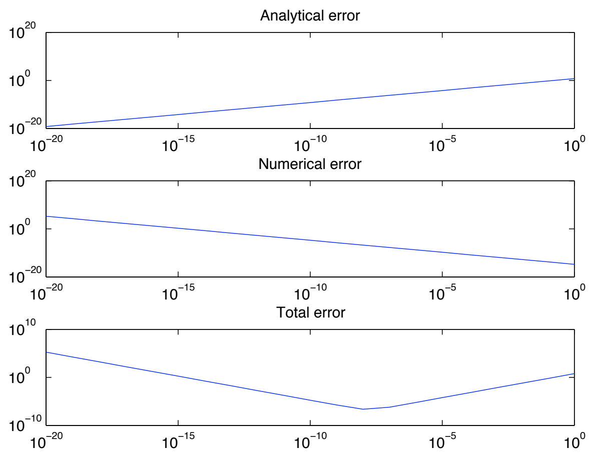


Figure 9.2.: Two sources contribute to the differentiation error: the analytical component ($h \to 0$) and the numerical component (loss of precision for $f(x) \approx f(x+h)$). The optimal value for h is ca. $10^{-8} \approx \sqrt{\texttt{eps}}$.

These errors are calculated for our example $f(x) = x^3$ and $x_0 = 2$ and plotted in Fig. 9.2.

To obtain the optimal h we calculate the first derivative of $\epsilon_A + \epsilon_N$ with respect to h and set it zero:

$$\frac{\partial}{\partial h}\left(\epsilon_A + \epsilon_N\right) = \frac{f^{(2)}(x)}{2} - \frac{\texttt{eps} f(x)}{f'(x)h^2} = 0 \tag{9.8}$$

$$h^* = \sqrt{\texttt{eps}}\sqrt{\frac{2f(x)}{f^{(2)}(x)f'(x)}} \tag{9.9}$$

NOTE: A rule of thumb is $h = \sqrt{\texttt{eps} \cdot f(x)} \approx 10^{-8}\sqrt{f(x)}$. This rule is based on the assumption that the expression $\frac{2}{f^{(2)}(x)f'(x)}$ is of order 1.

9.3.1. Higher order derivatives

Analytically, we obtain higher order derivatives by differentiating the derivative

$$f^{(n)}(x) = \frac{d}{dx}f^{(n-1)}(x) \tag{9.10}$$

Calculating higher order derivatives numerically is a challenging problem, as each further

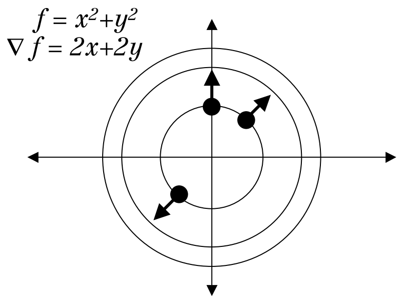


Figure 9.3.: The gradient points to the direction of the steepest incline.

differentiation adds to the error. If we evaluate (9.10) for $n = 2$ (second derivative) and insert (9.15), we obtain an explicit formula for the second derivative:

$$f''(x) = \frac{f(x+h) - 2f(x) + f(x-h)}{h^2} \tag{9.11}$$

In practice, this approach usually yields too large errors, see section 9.4.

9.3.2. Multivariate case

Partial derivatives pose no special problem to numeric differentiation. It makes no difference in applying (9.1) whether $f(\cdot)$ only depends on one x or on multiple variables.

Mixed second derivatives are obtained from the total differential:

$$\frac{\partial^2 f(x_0)}{\partial x_i \partial x_j} = \frac{f(x_0 + h_i e_i + h_j e_j) - f(x_0 - h_i e_i + h_j e_j) - f(x_0 + h_i e_i - h_j e_j) + f(x_0 - h_i e_i - h_j e_j)}{4 h_i h_j}$$

Where x_0 is a vector and e_i is the $i-$th unit vector, i.e. a vector with all zeros except the $i-$th element being 1. Also note that generally $h_i \neq h_j$

Definition 13 (Gradient). The gradient of a function $f(x) : \mathbb{R}^n \to \mathbb{R}$ is a vector of its partial derivatives. It is a vector that points into direction of the (locally) steepest incline of the function, see Fig. 9.3.

$$grad\ f(x_1, x_2, \ldots, x_n) = \nabla f = \begin{pmatrix} \frac{\partial f}{x_1} \\ \frac{\partial f}{x_2} \\ \vdots \\ \frac{\partial f}{x_n} \end{pmatrix} \tag{9.12}$$

The gradient is calculated component by component.

Definition 14 (Jacobian matrix). The Jacobian matrix of a function $f(x) : \mathbb{R}^n \to \mathbb{R}^m$ is a matrix of its first partial derivatives. It is equal to the gradient vector for scalar functions and is generally not square.

$$J_f = \nabla f = \begin{pmatrix} \frac{\partial f_1}{\partial x_1} & \frac{\partial f_2}{\partial x_1} & \cdots & \frac{\partial f_m}{\partial x_1} \\ \frac{\partial f_1}{\partial x_2} & \frac{\partial f_2}{\partial x_2} & & \frac{\partial f_m}{\partial x_2} \\ \vdots & & \ddots & \vdots \\ \frac{\partial f_1}{\partial x_n} & \frac{\partial f_2}{\partial x_n} & \cdots & \frac{\partial f_m}{\partial x_n} \end{pmatrix} \tag{9.13}$$

Definition 15 (Hessian matrix). The Hessian matrix of a function $f(x) : \mathbb{R}^n \to \mathbb{R}$ is a square matrix of its second partial derivatives:

$$H_f = \Delta f = \begin{pmatrix} \frac{\partial^2 f}{\partial x_1^2} & \frac{\partial^2 f}{\partial x_1 \partial x_2} & \cdots & \frac{\partial^2 f}{\partial x_1 \partial x_n} \\ \frac{\partial^2 f}{\partial x_2 \partial x_1} & \frac{\partial^2 f}{\partial x_2 \partial x_2} & & \frac{\partial^2 f}{\partial x_2 \partial x_n} \\ \vdots & & \ddots & \vdots \\ \frac{\partial^2 f}{\partial x_n \partial x_1} & \frac{\partial^2 f}{\partial x_n \partial x_1} & \cdots & \frac{\partial^2 f}{\partial x_n \partial x_n} \end{pmatrix} \tag{9.14}$$

This matrix is symmetric if $f(\cdot)$ is a continuous function. If it is positive definite, the function is at a minimum. When calculating the Hessian matrix element by element, it may happen that numerically $\frac{\partial^2 f}{\partial x_i \partial x_j} \neq \frac{\partial^2 f}{\partial x_j \partial x_i}$.

Example 21 (BHHH-Estimator.). Under some regularity assumptions, the maximum likelihood estimator

$$\hat{\theta} = \arg \underset{\max}{\theta} \sum l_t(\theta)$$

in a time series environment is normally distributed as $\sqrt{T}(\hat{\theta} - \theta) \overset{d}{\to} N(0, \Delta^{-1})$, with the Fisher information matrix $\Delta = -E\left[\frac{1}{T}\sum \frac{\partial^2 l_t(\theta)}{\partial\theta\partial\theta'}\right]$. This sum of Hessian matrices is numerically intensive to calculate and not guaranteed to be positive definite.

The BHHH-estimator overcomes this in a maximum likelihood setting by estimating the Fisher Information matrix as

$$\hat{\Delta} = \frac{1}{T}\sum \left(\frac{\partial l_t}{\partial \theta}\right)\left(\frac{\partial l_t}{\partial \theta}\right)'$$

This sum of outer products of gradients is much faster to calculate and guaranteed to be positive definite. This estimator is consistent, but slightly less efficient, i.e. a few additional iterations are the price for a more stable algorithm.

9.3.3. Improved numerical differentiation schemes

There are two routes to improve numerical differentiation: (1) higher-order algorithms and (2) polynomial approximation.

9.4. Higher order algorithms

One way to reduce the analytical error is to use a higher order (Taylor) approximation. The simplest improvement is the central difference method:

$$f'(x) \approx \frac{f(x+h) - f(x-h)}{2h} \tag{9.15}$$

this cancels out the first term in (9.5). The calculation of the optimal h becomes

$$\frac{\partial}{\partial h}\big(\epsilon_A + \epsilon_N\big) = \frac{4hf^{(3)}(x)}{3! \cdot 2^3} - \frac{\texttt{eps} f(x)}{f'(x)h^2} = 0 \tag{9.16}$$

$$h^* = \sqrt[3]{\texttt{eps}}\,\sqrt[3]{\frac{12f(x)}{f^{(3)}(x)f'(x)}} \tag{9.17}$$

The rule of thumb for h is now larger: $\sqrt[3]{\texttt{eps} \cdot f(x)} \approx 6 \times 10^{-6}\sqrt[3]{f(x)}$.

NOTE: The central difference method is certainly more accurate, but forward differences may still be preferable in some cases, especially when calculating the gradient or the Jacobian matrix (see next section).

Even higher order differentiation schemes include:

$$f'(x) \quad\approx\quad \frac{1}{12h}\Big(f(x-2h) - 8f(x-h) + 8f(x+h) - f(x+2h)\Big) \tag{9.18}$$

Differential quadrature

An alternative is to approximate $f(\cdot)$ by a polynomial (or another set of basis functions) and to perform a symbolic differentiation. This is especially useful when we differentiate noisy data and want to ensure certain known properties (like positivity of the first derivative).

NOTE: Do not mix up the following:

- *Higher order derivatives* $\frac{d^n}{dx^n}f(x)$, e.g. the second derivative for $n = 2$
- *Higher order algorithms* are algorithms that use a better-than-necessary Taylor approximation. For the derivative, this means they use more than $n+1$ function evaluations to obtain $\frac{d^n}{dx^n}f(x)$.

NOTE: It is not without reason that polynomial or functional approximations are widely used as an alternative to numerical differentiation. Especially when working with data, numerical differentiation is often too noisy.

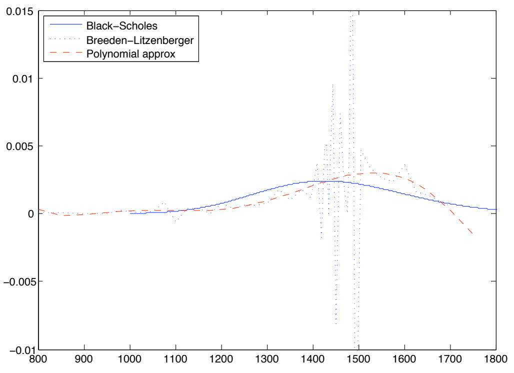


Figure 9.4.: Comparing the Breeden-Litzenberger formula to a symbolic differentiation of a 8-th order polynomial and the Black-Scholes formula.

9.4.1. Higher precision and adaptive derivatives

See the DERIVESTsuite, available at `http://www.mathworks.com/matlabcentral/fileexchange/13490-adaptive-robust-numerical-differentiation`. This suite also comes with an excellent, 9-page pdf documentation.

9.5. Numeric differentiation of data

Given two vectors $x = (x_1, x_2, \dots x_n)'$ and $y = (f(x_1), f(x_2), \dots f(x_n))'$, there are two ways to calculate the derivative by applying (9.1) and (9.15):

$$\begin{array}{lll} f'(x_m) = \frac{y_{i+1}-y_i}{h} & x_m = \frac{x_i+x_{i+1}}{2} & h = x_i - x_{i+1} \\ f'(x_m) = \frac{y_{i+1}-y_{i-1}}{h} & x_m = \frac{x_{i-1}+x_{i+1}}{2} & h = x_{i+1} - x_{i-1} \end{array} \tag{9.19}$$

Note that data is usually spaced quite wide, so that $h \gg h^*$. This implies that the numerical error is of no concern. The first equation produces derivatives at points in the middle of the original grid, the second produces points at the same grid points, but at a lower precision. In any case, the vector of derivatives is shorter (by one or two) than the original one.

Example 22 (The formula of Breeden and Litzenberger (1978)). A famous result is the relation between risk-neutral probability density and option prices:

$$q(K) = \frac{\partial^2 C}{\partial K^2} \tag{9.20}$$

We compare three different approaches. First a double numeric differentiation of option prices, second a symbolic differentiation of a fitted 8-th order polynomial and last a Black-

Scholes approximation. MATLAB code: `ndiff_breelitz.m` .

9.6. PC-Lab: Higher order differentiation algortithms

- `differentiation_higher_order.m`
- `differentiation_higher_order_with_noise.m`

9.7. Exercises

Exercise 9.1. Reproduce the figure 9.2 for (a) $f(x) = e^x$; (b) $f(x) = \ln(x)$; (c) $f(x) = x^{10}$; each for $x_0 = 5$.

10. Optimization

References: Mathews and Fink, section 8; Brandimarte section 3; Judd, section 4; Moler, Kharab and Guenther (general), Chong and Zak (advanced), Gilli, Maringer and Schumann, sec 10-13,

Example 23. When we considered OLS, we wanted to find the "best" estimator for the regression parameters. This problem has two steps. The first part is the question: "What means best"? In a regression with N observations, there are N error terms ϵ_i. The "best" estimator, however, can only be defined by *one criterion.* We need to find a way to combine these N error terms. In the case of OLS, we have good reasons to choose the sum of all squared error terms, $SSE = \sum_i \epsilon_i^2$ as our criterion. We can think of the SSE as a distance from the optimal (theoretical) case with all zero errors. A criterion that attaches a scalar number to a generalized distance is called *metric.*

The second step seems to be straight forward: we need to minimize (or maximize, depending on the definition) our criterion. The next section focuses on choosing the right metric, while the remainder of this chapter is devoted to different techniques for finding a minimum.

10.1. Metrics

The concept of "distance" seems to be quite natural and it seems that there is not a lot of room for interpretation. However, even classical distances can be defined in different ways, as Fig. 10.2 shows: While the standard distance in the two-dimensional space is just the shortest line connecting two points, French railway passengers must frequently get from A to Paris and from Paris to B, as there are very few trains that cross the country. Equally, pedestrians in Manhattan can only walk along streets and avenues, making the walking distance much longer than the distance measure in $\mathbb{R}^2$.

In the econometrics context, a metric is a measure for the "distance" between model and data, between $\hat{\mathbf{y}}$ to $\mathbf{y}$, where $\hat{\mathbf{y}} - \mathbf{y}$ is an N-dimensional vector with N, the number of observations, usually large. Depending on the problem, one is quite free to choose a metric of one's own, as the example of least squares versus least absolute deviations shows. There are, however, four criteria, that a metric must meet.

Definition 16. A metric is a function $d(i,j) : \mathbb{R}^n \times \mathbb{R}^n \rightarrow \mathbb{R}$ of two points (i,j) in $\mathbb{R}^n$, which fulfills the following four criteria:

1. Non-negativity: $d(i,j) \geq 0$
   Any distance is positive or zero.
2. Identity of indiscernibles: $d(i,j) = 0 \Leftrightarrow i = j$
   The distance between two points is only zero, iff they are identical.
3. Symmetry: $d(i,j) = d(j,i)$
   The distance from i to j as the same as the distance from j to i.
4. Triangle inequality: $d(i,k) \leq d(i,j) + d(j,k)$
   The direct path between two points is the shortest

10.2. Canonical formulation of the optimization problem

Finding the minimum, the optimum or the maximum refers to the same problem. Instead of maximizing $f(x)$, one can equally minimize $-f(x)$. As already mentioned, optimizing is either minimizing a measure of disutility(i.e. cost, error, ...) or maximizing a measure of utility (i.e. profit, quality, ...). To simplify the exposition, we follow the literature and discuss only the minimization problem.

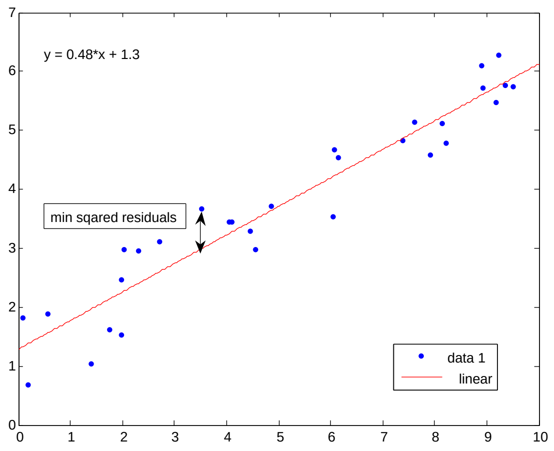


Figure 10.1.: The error term in OLS can be interpreted as distance between data and model.

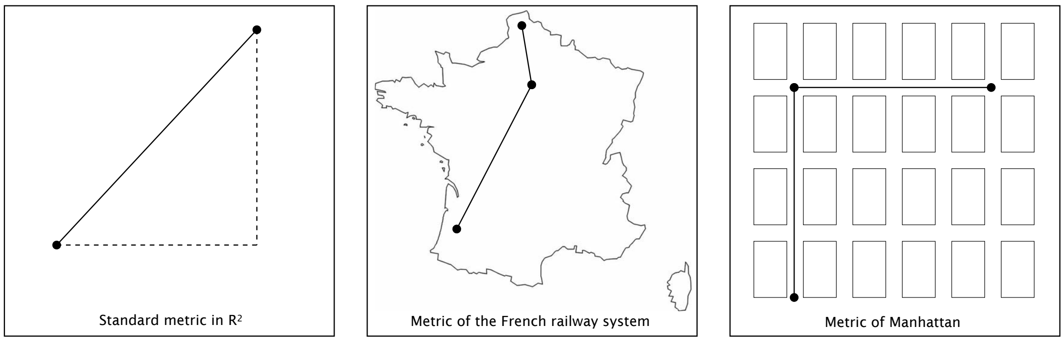


Figure 10.2.: Different concepts of distance, which all fulfill the definition of a metric.

10.2.1. The static minimization problem

A minimization problem is a problem of the type

$$\min_{\theta} f(\theta, X) \qquad f: \mathbb{R}^n \to \mathbb{R}, \quad \theta \in \Theta \subset \mathbb{R}^m \tag{10.1}$$

where $f(\cdot)$ is called *objective function* and Θ is called *feasible set*. Some examples for objective functions are given in Tab. 10.1

10.2.2. A few facts about minima

First and second derivative. If a function is twice differentiable, a (local) minimum is found at x, where

$$\begin{array}{llll} \text{1-dim:} & f'(x) = 0 & \text{and} & f''(x) > 0 \\ n\text{-dim:} & \frac{\partial f}{\partial \mathbf{x}} = 0 & \text{and} & \frac{\partial^2 f}{\partial x_i \partial x_j} \quad \text{(Hessian pos. definite)} \end{array} \tag{10.2}$$

The optimization problem can be related to finding the root of the first derivative. Some optimization algorithms actually use this approach, but it is not always practical for two reasons. First, the equation $f'(x) = 0$ may not be straight forward to solve. Second, if $f'(x)$ is not known analytically, one would have to resort to numerical differentiation, which is computationally costly and plagued with precision problems.

mean-variance utility
least squared errors
social welfare function

Table 10.1.: Some objective functions

Existence of minima. For a large class of functions, we are even guaranteed that there exists a minimum: by the *Weierstrass extreme value theorem*, we know that any continuous function on a closed interval $[a, b]$ has both a minimum and a maximum value in this interval.

Minimum preserving transformations. Any transformation using a strictly increasing function preserves the location of the minimum. Any affine transformation ($\tilde{y} = \alpha + \beta \cdot y$) with $\beta > 0$ is an example for a strictly increasing transformation. In practise, this means it does not matter whether we minimize the sum of squared errors, the mean squared error or the root mean squared error – all these metrics will produce the same parameter vector $\hat{\theta}$. (We obtain, however, different results if we use mean absolute deviations or the mean squared relative error, because they are not obtained by a strictly increasing transformation).

Minimum preserving transformations are also used to *regularise* optimisation problems, i.e. to flat objective functions that are difficult to minimise into functions with the same minimum that are easier to handle for optimisation algorithms.

10.2.3. Classification of minimization algorithms

Different algorithms have different requirements in terms of the form of the objective function and our knowledge about it. We can organize them along several lines:

By convexity. The most important distinction: as the name suggests, convex algorithms work only for convex functions, while non-convex ones do not set this condition.

By use of the derivative be it explicitly (functional form of the derivative known) or implicitly (derivative calculated numerically).

By starting values. Single starting point (most algorithms) or starting interval (e.g. grid search, genetic algorithms)

By type of algorithm. Deterministic or stochastic. Iterative or not.

There are also two-step algorithms that combine some of the above features. Beware that there is no "one size fits all" minimization algorithm, each of the following algorithms has its strengths and weaknesses and works only under certain conditions. It is a major task of the researcher to choose the best applicable algorithm.

10.3. Convex optimization

Most classical optimization algorithms and many popular MATLAB minimization commands require the function under consideration to be convex. Convexity is a very restrictive assumption, but it ensures the existence of a single, global minimum which permits the creation of very efficient optimization routines. It is, however rarely met apart from some theoretical case. While there are abundant examples in the literature of an unwarranted use of convex optimization, it has its merits, as any function is locally convex around its minimum. Given good starting values, convex algorithms yield fast and correct results.

10.3.1. Golden search

This algorithm works pretty much like bisection, just that the interval is not cut in half, but divided according to the golden ratio ($\frac{\sqrt{5}-1}{2}$). This very simple algorithm works only in one dimension.

Golden Search
Convex, derivative-free, deterministic, not parallel

Advantages	**Disadvantages**
+ Fast convergence	– Can be fooled by local minima
+ Can handle discontinuity	– Need good guess for starting range
+ Arbitrary precision	
+/- Used for problems that have been reduced to one dimension	

10.3.2. Direct search

One of the most famous direct search algorithms or n-dimensional minimization of convex functions is the Nelder-Mead simplex algorithm. The idea is simple: try a few points, discard the worst one and replace it by a better one. Repeat until the target precision has been achieved. An example in 2D can be found at `tinyurl.com/28ro3c`. The algorithm for n-dimensional functions looks like this:

1. Start with $n + 1$ points around the starting value x_0, and evaluate the objective function at each point.
2. The worst point is the one with the highest value of the objective function. This point will be replaced.
3. Calculate the *distance vector* from the deleted point to the center of gravity of the remaining points. Now three candidate points for the replacement are formed:
   - **Mirror point.** Point the distance vector from the center of gravity of the remaining points (i.e. away from the worst point).
   - **Expansion point.** Just like the mirror point, but move twice the length of the distance vector. This will increase the size of the point cloud.

- **Contraction point.** Just like the mirror point, but move half the length of the distance vector. This will shrink the size of the simplex.

4. Take the best of the three candidate points. Repeat (2-4) until the size of the point cloud is smaller than the precision required.

Interpretation. The interplay of expansion and contraction points allows for a variable step size: increasing swiftly far away from the minimum and decreasing close to the minimum to enhance the algorithm's precision.

`MATLAB note`. The Nelder-Mead algorithm is implemented in MATLAB's `fminsearch` command `fminsearch(function_handle, first_argument, options)`

`Octave note`. There is no `fminsearch()`. Instead, maximize $-f(x)$ using `mdsmax()`.

Nelder-Mead Simplex Algorithm
Convex, derivative-free, deterministic, parallel up to # of candiate points

Advantages	**Disadvantages**
+ Don't need derivative	– Can be fooled by local minima
+ Can handle discontinuity	– Need good guess for starting values in presence of local minima
+ Very fast, good scaling	
+ Arbitrary precision	

10.3.3. Gradient-based methods: steepest descent (a quasi-Newton method)

The idea is simple and convincing: if you always walk uphill you will eventually reach the mountain top. Conversely, going downhill will lead you to the lowest point of the valley – the minimum. The best way to do this is to follow the direction of the steepest incline/descent. The direction of the steepest incline is given by the gradient ∇f, the steepest descent is consequently $-\nabla f$, see Fig. 9.3.

The steepest descent algorithm is simple:

1. Start with one point x_0, calculate the gradient
2. Subsequent points are obtained by subtracting a multiple of the gradient: $x_{n+1} = x_n - s\nabla f(x_n)$. Perform a grid search to find a locally optimal value of s.
3. Repeat (2) until $f(x_{n+1})$ and $f(x_n)$ are within the desired tolerance (i.e. $\nabla f(x_n)$ is close to zero – near a minimum/maximum)

The optimal value for s is found in a two-step procedure. We know that a very small and a very large number for s are not optimal (to understand why, think of the consequences of $s \to 0$ and $s \to \infty$. Step 1 (bracketing): We start with an arbitrary interval for $s \in [a, b]$, we try smaller numbers for $a(a/2, a/4, a/8, \dots)$ and larger numbers for $b(2b, 4b, 8b, \dots)$ until the result gets *worse* by expanding the interval. We now know that the optimal s must be in the interval. Step 2: perform a bisection to find the optimal value of s by evaluating $f(x_n - s\nabla f(x_n))$.

Interpretation. The gradient shows the direction, but how far should we go? The line search part in the algorithm draws a line along the gradient's direction and looks for the optimal point along this line. This reduces the problem to a sequence one dimensional searches, which are very fast.

Steepest descent
Convex, derivative-based, deterministic, parallel up to # of variables

Advantages	**Disadvantages**
+ Very fast	– Need explicit gradient or costly numerical calc.
+ Intuitive algorithm	– Cannot handle discontinuities
+/– First order approximation: fast, simple, not precise	

> `MATLAB note`. Steepest descent is implemented in the command `fminunc`, albeit not in the default version. If an analytical gradient is supplied, `fminunc` uses it, otherwise it calculates the gradient numerically.

More on trust region methods:
`http://www.applied-mathematics.net/optimization/optimizationIntro.html`

10.3.4. Advanced gradient-based methods: using the second derivative

The steepest descent is in fact a local fist-order approximation of the function. One way to improve it is to include the second derivative (the Hessian matrix). This allows for a more precise approximation that may be valid in a larger *region* around the current value, thus permitting larger steps towards the minimum. This efficiency comes at a price: we need to (numerically) calculate the second derivative. A whole class of so-called *Quasi-Newton methods* is devoted at finding fast approximations for the Hessian matrix.

MATLAB's `fminunc` and `lsqnonlin` are examples for these algorithms.

10.4. Nonconvex optimization

10.4.1. Grid search

This algorithm is sometimes called a "brute force" algorithm, not without reason. The idea is find the minimum by trying all possible values. This leads us directly to the dilemma of this method: trying all possible values will guarantee us to find the minimum, even in the absence of any knowledge of the objective function, but this would take an infinite amount of time. We have to define a grid of numbers that we want to try, which limits our search to an interval and reduces the precision. Furthermore, if we want to optimize an n-dimensional function, a tenfold increase in precision requires an increase in the number of grid points by a factor of 10^n. This phenomenon is called the *curse of dimensionality.*

The algorithm is straight forward:

1. For every variable define an interval and a number of grid points (can be different in every dimension).
2. Produce a regular grid of points within the interval(s). Evaluate the objective function at each grid point and make a list of the results.
3. Search for the lowest result in the list – this is the minimum.

Discussion. The strengths and weaknesses of grid search lie in the fact that it makes no assumptions about the function (except the search interval). It cannot be misled by wrong assumptions, but it cannot increase its efficiency by using such a knowledge. Furthermore, it is the only non-iterative algorithm, which limits its potential precision. One could devise an iterative version of grid search (search a smaller grid around the best point in the second iteration), but this would imply making an assumption – namely smoothness – about the function.

Example: grid search
```
ff=@(x) x.^ 2 - x + 2;
x0 = linspace(-10,10,1000);
ytry = ff(x0);
pos = find(ytry==min(ytry));
min = x0(pos);
```

Grid search
Non-convex, derivative-free, deterministic, fully parallel

Advantages	**Disadvantages**
+ Works with all functions	– Slow convergence, bad scaling with precision
+ Easy to implement	– Curse of dimensionality (de facto limited to 2)
	– Requires starting interval

10.4.2. Combined search

Combined search is a widely used method that draws on the strengths of grid search and direct search. The idea is to perform a grid search on a rather wide grid and to use the (few) best grid points as starting values for a direct search. This draws from the fact that every function is locally convex around its minimum. The implicit assumption in this method is that the convexity region around the minimum is larger than the spacing of the grid points. If this is the case, combined search is the perfect method for problems in which we have little knowledge, yet require high precision. We start by only defining the grid and end up with arbitrary precision results form direct search.

The algorithm:

1. Perform a grid search on a comparatively wide grid.
2. Identify the n best grid points (n usually 5-10).
3. Perform n direct search operations with the best points from (2) as starting points.
4. Take the best result from (3) as minimum.

Combined search
Combined method, parallel up to # of direct searches

Advantages	**Disadvantages**
+ The best of both worlds	– Computationally intensive

10.5. Stochastic optimization

10.5.1. Differential evolution

Differential evolution (DE) is based on the principles of evolution, i.e. selection and mutation. Contrary to most other algorithms, a whole *population* of trial vectors is tacked from one step to the next one.

The basic version of the algorithm is:

1. Define a starting interval for each variable.
2. Start with a population of n vectors, randomly chosen out of the starting interval. The population size is usually several times (2-10) the number of variables. Evaluate the objective function for each population member.
3. **Mutation phase:** for each vector x_i, find a trial vector $\tilde{x}_i$ by random mutation:
   $\tilde{x}_i = x_i + \delta(x_k - x_l), \qquad i \neq k \neq l, \quad 0 < \delta < 2$
4. **Selection phase:** compare the value of the objective function for each x_i and $\tilde{x}_i$ and retain the vector with the better value. Continue with (3) until the objective function is below a threshold or the maximum number of steps (100 to 1000) is reached.

There are different variations of this algorithm that use more sophisticated mutation schemes, e.g. mutating each component of a vector independently, adding some random noise to the mutation or forcing at least one component of the vector to remain unchanged.

DE is not very precise, but this is outweighed by its speed and flexibility. Again, we can employ a combined strategy to use the DE result as input for a direct search.

Discussion. The success of DE depends on choosing a good initial range for each variable and to maintain a sufficiently large population. An interesting feature is that as the population converges towards a minimum, the mean distance $||x_k - x_l||$ decreases, making the steps smaller and increasing the precision. In most practical cases, the precision of DE increases only very slowly, thus lending hand to a combination of DE and direct search.

Differential evolution
Non-convex, derivative-free, stochastic, parallel up to # of population members

Advantages	**Disadvantages**
+ Works with all functions	– Computationally intensive
+ Comparatively fast	– Requires initial interval
	– Can be (rarely) fooled into local minima
	– Low precision ($\rightarrow$ combine with direct search)

10.5.2. Simulated annealing

In a nutshell, simulated annealing combines the Monte Carlo method with direct search. This method, which was pioneered in solid state physics, solves the problem of local minima by assigning a probability p to jump into "wrong" direction. This makes it possible to jump out of local-minimum traps. The probability p is slowly reduced from 1 to 0 (in physics-speak "cooled"). In the last stage, simulated annealing behaves pretty much like direct search and homes in on the global minimum.

Simulated annealing
Non-convex, derivative-free, stochastic, not parallel

Advantages	**Disadvantages**
+ Works with all functions	– Extremely slow
+ Guaranteed to find minimum	– Choice of "cooling parameter" not clear
+/– Good starting parameters still help a great deal	

| See `sim_anneal.m`

10.6. Constrained optimization

A constrained optimization problem is a problem of the type:

$$\min f(x, y) \quad s.t. \quad x + y > 0$$

Applications: budget constraints, economic constraints (e.g. positive interest rates), model/legal constraints (e.g. no short selling), econometric constraints (e.g. stability of time series process).

There are two ways how to implement a constrained optimization, and both have their distinct advantages and disadvantages.

10.6.1. Implementing constraints in the optimization algorithm

MATLAB provides some algorithms for constrained optimization such as `fmincon()`, which is gradient-based. Furthermore, a constrained version of the Nelder-Mead algorithm called `fminsearchbnd()` can be downloaded from `matlabcentral.com`. These functions allow the implementation of individual constraints, such as minima and maxima for each variable or joint constraints of the form $a'x < 0$, where a is a parameter vector and x is the vector to be optimized. The functional form of constraints is, however, limited to a few simple cases.

Constrained optimization with individual constraints is also available for differential evolution and simulated annealing.

NOTE: To allow unconstrained optimization for certain elements of a vector, use ± `Inf` as constraints.

```
fmincon(FUN,X0,A,B,Aeq,Beq,LB,UB,NONLCON)
    X0 = starting value
    A*X<=B
    Aeq*X==Beq
    LB < X < UB (element per element)
    NONLCON = function that produces C, Ceq with
        C(X) <= 0
        Ceq(X) = C(X)
    use [ ] to skip a constraint
```

NOTE: Using the following identities ...

1. $c'\mathbf{x} = max \Leftrightarrow -c'\mathbf{x} = min$ (under the *same* conditions)
2. An inequality $a'\mathbf{x} \geq b$ can be written as an equality using a slack variable $y > 0$: $a'\mathbf{x} - y = b$
3. An equality $a'\mathbf{x} = b$ can be replaced by two inequalities $a'\mathbf{x} \geq b$ and $-a'\mathbf{x} \geq -b$.
4. Any component x_i of $\mathbf{x}$ which can be positive or negative can be replaced by $x_i = x_i^+ - x_i^-$ with $x_i^+, x_i^- \geq 0$.
5. The inequality $a'\mathbf{x} \leq b$ can be transformed to $-a'\mathbf{x} \geq -b$ by multiplying the LHS and RHS by -1.

Constraint	transformation
$x > 0$	$x = y^2$
$x > \alpha$	$x = y^2 + \alpha$
$x \in [-\pi/2, \pi/2]$	$x = atan(y)$
$x_2 > x_1$	$x_1 = y_1;\ x_2 = y_1 + y_2^2$

Table 10.2.: Implementing constraints for $f(x)$ via transformations. The unconstrained variable is y.

... every linear optimization problem can be formulated as $\min_x c'\mathbf{x}$ s.t. $Ax = b$ and $\mathbf{x} \geq 0$.

10.6.2. Implementing constraints through transformations

One approach to implementing constraints in an optimization is to find a transformed model, that can be optimized without constraints. For example, to ensure that x is positive, one could use the transformation $x = y^2$, with y unconstrained. Such a transformation needs to be injective and – depending on the optimizer – continuous (the latter excludes all type of *modulus*). A list of common transformations is shown in Tab. 10.2.

10.6.3. Practical considerations

Constrained vs. unconstrained optimization. Can we identify truly unconstrained cases, i.e. cases in which *all* conceivable parameter combinations would yield a sensible result? In fact, this will rarely be the case. Think of an affine model (be it term structure, portfolio analysis or option pricing). The mean reversion and vol-of-vol parameters make only sense within a certain range, the FFT of the model may even give NaN results for some combinations. Thus, it may be a good idea to constrain some parameters to *realistic values*, even if the problem itself is unconstrained, as trust-region algorithms may try really weird combinations of parameters.

Matrix-valued arguments. Standard algorithms cannot handle matrix-valued arguments, but this is usually not a problem. Transform the matrix into a vector using `A(:)` and optimize over the vector. Use `reshape(v,m,n)` to create a $m \times n$ matrix from vector v. Notice that *all* optimization parameters need to be fused into one parameter vector for optimization.

Treatment of NaNs. As optimization algorithms try all sorts of parameter combinations, it may occur that they run into a range where the objective function produces a `NaN` (not a number). The standard optimization algorithms in MATLAB cannot manage this situation and may mistakenly prefer the parameters that produce the `NaN`. An easy way out of this dilemma is to add a check for NaNs at the end of the objective function:

```
if isnan(loss)
  loss = 1E10;
end
```

Random numbers in the optimiser. By their nature, stochastic algorithms will provide different solutions when applied repetitively to the same problem. In most cases, the differences will be insignificant, but even stochastic algorithms can fall into a local minimum trap. It is highly likely that they will do so in only a fraction of the runs. Re-running a stochastic optimiser is thus a check for the presence of local minima or other problems in the form of the objective function. Keep in mind, however, that stochastic optimisers depend on the state of the random number generator and of its seed, just like any other algorithm that depends on random numbers. Immediatley after starting MATLAB, the random number generator will always be in the same state and a stochastic optimiser will always produce exactly the same results.

Random numbers in the objective function. If random numbers are used to calculate the value of the objective function (e.g. if the objective function is based on some Monte Carlo simulation), its values will fluctuate even if the argument is not changed. Under such circumstances, even a well-behaved objective function ceases to be continuous: infinitely small changes of the input arguments may result in considerably large changes of its value. This is not desirable for many optimisation algorithms. The Nelder-Mead algorithm, for example, uses the volume of the simplex as a stopping criterion. Adding a point close to the optimum my in fat increase the size of the simplex, worse, close to the optimum the size of the simplex will become stochastic. In this case it is important to fix the seed of the random number generator for every evaluation of the objective function. This freezes the noise created by the randomisation and guarantees local continuity of the objective function.

10.7. PC lab: Fooling convex optimization

`diary('opt.txt')`	Don't forget to start a diary.

Using and fooling fminsearch

`f=@(x) x.^ 2 - x + 2;`	A simple, convex function
`fplot(f,[-2 2])`	A plot for convenience
`[x,fval,eflag,outp]=fminsearch(f,2)`	
	Start at $x = 2$. Results: minimum x_0, $f(x_0)$, exitflag (1=found, 0=aborted), information on the algorithm.
`f=@(x)sin(x)+0.2*sin(10*x)`	Nasty function
`fplot(f,[0,2*pi])`	...with many local minima
`x0=2;`	First starting value
`x_min=fminsearch(f,x0)`	We find some local minimum. (Use 2x cursor-up)
`x0=2.7;`	Second starting value
`x_min=fminsearch(f,x0)`	Find another local minimum, not even the closest one.
`x0=2.9;`	Third trial.
`x_min=fminsearch(f,x0)`	

Conclusion: Nelder-Mead can easily be fooled → Do not let `fminsearch` fool you!

Fminsearch and a two-dimensional function

`f=@(x,y)(x-1).^2+(y+1).^2`	A nice hyperbola
`[X,Y]=meshgrid(-8:.5:8);`	Prepare for a 3D plot
`Z=f(X,Y);`	Calculate all z-values
`mesh(X,Y,Z)`	Nice 3D plot
`surfc(X,Y,Z)`	Another one – note the isolines on the xy-surface.
`fminsearch(f,[0,0])`	Let us try to minimize f – we get an error message.
`f=@(v)(v(1)-1)^2+(v(2)+1)^2`	Rewrite in vector format: $x = v(1); y = v(2)$.
	Note: meshgrid and 3D plotting require vector compatible functions, fminsearch does not.
`fminsearch(f,[0,0])`	Now it works and we get a vector result.

Fminunc with explicit gradient

First create a MATLAB function that supplies the function value and the gradient.

```
function [f,g] = myfun(x)
   f = x(1)^2 + x(2)^2;       Our function
   g = [2*x(1); 2*x(2)];      Its gradient (note: gradient is a column vector)
end
```

Now we can minimize it

```
options = optimset('GradObj','on');
fminunc(@myfun,[1;1], options)
```

10.8. PC lab: Stochastic optimization

- Run `sim_anneal.m` in cell-mode.
- Download the MATLAB demo program for differential evolution form `www.icsi.berkeley.edu/~storn/code.html` and run it.

10.9. Exercises

Exercise 10.1 (Utility)**.** The utility of an agent is given as $u(K) = K - 0.05K^2$, where K can take values in the range $[0 \dots 100]$. (a) Maximize u using `fminsearch()`. (b) Calculate 1000 values of u. Find K so that u has the maximal value.

Exercise 10.2. The utility of an agent is given as $u(K, L) = K - 0.05 * K^2 + log(L) - 0.25 - (K - 20)^2 + (L - 20)^2$, where both K and L can take values in the range $[0 \dots 100]$. (a) Maximize u using `fminsearch()`. (b) Calculate 1000×1000 values of u. Find K, L so that u has the maximal value. Produce a 2D and a 3D plot. (c) Add the budget constraint $K + L < 20$ and optimize using `fmincon()`. Is the budget constraint binding? (Answer this question in one line in the comment)

Exercise 10.3. Two-dimensional minimization:
(a) Minimize $f(x, y) = 3x^2 + 2xy + y2 + 3$ using fminsearch.
(b) Minimize the function in (a) using fminunc. Choose the version of fminunc in which you provide the gradient.

Exercise 10.4. Minimize$f(x) = 2x^2 + 2x + 2$ using a (simplified version of the) Nelder-Mead algorithm. Program the algorithm yourself and do not use any optimization functions from MATLAB. It is not necessary to program a general version of the algorithm. A 1-dimensional version is perfectly sufficient. Choose a minimum precision of 10^{-4} for the value of the function.

Exercise 10.5. In section 8.4.2 we used bisection to calculate the implied volatility. Could we have used a Nelder-Mead simplex algorithm?

(a) Write a small program that calculates the implied volatility using `fminsearch`.

(b) Compare the execution speeds of `fminsearch` and bisection.

(c) Write about 150 words in the comments about the differences in preliminaries, convergence speed and the necessary knowledge about $f(x)$ between bisection and the Nelder-Mead simplex algorithm.

Exercise 10.6 (Minimizing functions)**.** Consider the function $f(x) = x^6 - 14x^5 + 62x^4 - 76x^3 - 63x^2 + 90x$.

(a) find a convenient and fast representation in MATLAB. This representation should not make use of the power operator `^` or the function `power()`. Produce a plot of the function on a suitable interval.

(b) minimize the function using `fminsearch`. Show that for two different starting values it is possible to obtain totally different results. Using the plot produced in (a), guess which range of starting values will lead to the global minimum.

(c) Minimize the function using a grid search over the interval $[-10, 10]$. Program the line search yourself and do not use any optimization functions from MATLAB. Show that for different numbers of grid points one obtains slightly different, but similar results.

(d) Minimize it using Differential Evolution. Use the code from `www.icsi.berkeley.edu/~storn/code.html`. You will have to change the objective function and a few parameters in the main program.

(e) Minimize it using simulated annealing. Base your program on `sim_anneal.m`. First, only change the objective function. If need be, change the parameters of the simulated annealing algorithm.

(f) Write a one-page summary of your findings, including (1) all starting values and results in a small table (2) a short interpretation of each result and (3) all changes that you made to the differential evolution and simulated annealing code. Submit only this summary in PDF format.

11. Numeric integration and transform methods

References: Mathews and Fink, section 7; Brandimarte, section 2.5; Judd, section 7; Moler, Kharab and Guenther (numerical integration only)

11.1. The Riemann integral

Integrals are key to calculating probabilities and expectations and therefore they appear everywhere in economics, from solving an agent's consumption problem to pricing assets. There are three main motives to apply numerical integration: (a) if there is no closed-form solution, (b) if the closed-form solution is too complicated or (c) if we want to calculate an integral of some data.

The numeric integral can be directly derived from the integral's definition as a limit of the Riemann sum:

$$\int_a^b f(x)dx = \lim_{N\to\infty} \sum_{k=1}^{N} f(x_k^*) \cdot h \tag{11.1}$$

with $h = \frac{b-a}{N}$ and x_k^* an arbitrary point in the interval $[h(k-1), hk]$. In the limit the choice of x_k^* within the interval irrelevant, because for $N \to \infty$ the width $h \to 0$. In a

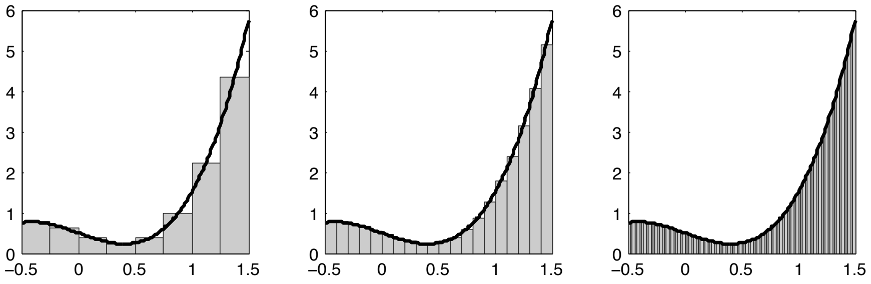


Figure 11.1.: Convergence of the Riemann sum for $f(x) = 2x^3 - x + 0.5$ on $[-0.5, 1.5]$ and x^* chosen in the middle of Δx. The areas of the rectangles are $2.4655(n = 8)$; $2.4949(n = 15)$ and $2.4998(n = 75)$. The exact value of the integral is 2.5.

graphical interpretation, the summands $f(x_k^*) \cdot h$ can be represented as rectangles of width h and the Riemann sum can be interpreted as the sum of their areas, see Fig. 11.1.

11.1.1. Two important properties of the integral

The integral – basically being the limit of a sum – has two important properties that are of great use for numeric integration.

Convergence. The sum in (11.1) converges to the integral regardless which point x_k^* we choose. On a similar line of argument, it can be shown that convergence is also assured regardless of how we approximate the function to be integrated.

Linear property of integration Just like a sum, the integral can be split at any point

$$\int_{x_0}^{x_2} f(x)dx = \int_{x_0}^{x_1} f(x)dx + \int_{x_1}^{x_2} f(x)dx \qquad \forall\ x_0 < x_1 < x_2$$

this means we can choose any combination of "slices" – even of different widths – when integrating a function. Many efficient integration algorithms are based on this simple property.

11.2. Numeric integration

The first step from Riemann integration to numerical integration, is to replace $n \to \infty$ with $n \to$ "many". The obvious question is, what value is sufficient for "many". Two things are clear by the convergence of the Riemann sum. First, the approximation error will decrease with N. Thus the choice of N depends on the target precision. Second, convergence implies there is no too large N. We search for the optimal N in order to reduce computation time. There are two approaches to find N: (1) an analytical approximation of the error (rarely used) and (2) an iterative approach (adaptive quadrature – the standard algorithm).

11.2.1. Midpoint rule: a very simple integration scheme

Approximating the integration error. An obvious choice for x_k^* is the middle of the interval (therefore the name "midpoint rule"). Any first order error is cancelled out. Only second or higher order errors remain. The result is exact for constant or linear functions. The leading error term is the second derivative. We can approximate the absolute integration error ε_N:

$$\varepsilon_N \propto N \cdot (dx)^3 \cdot f''(x) = \frac{(b-a)^3}{N^2} \cdot f''(x) \tag{11.2}$$

In many cases, we can make sensible assumptions on the bounds of $f''(x)$, thus give an upper bound for the integration error and calculate and N^* such that the overall error is below some margin.

Practical implications and rule of thumb. The midpoint rule is the simplest of all numeric integration schemes. It should only be used where a rough result is enough or where checking the results is easily possible. A simple check is to compare the integration results for N and for $10 \cdot N$. A good starting point is $N = 1000$. Note that the midpoint rule is the least efficient integration scheme. Using a more sophisticated one will greatly reduce the number of evaluations of $f(\cdot)$.

Example: a hand-made midpoint rule integral [0,1]

`f=@(x)x.^2+x;`	Note the function is vector-compatible.
`dx=0.1;`	Choice of Δx
`x=dx/2:dx:(1-dx/2);`	Midpoints start at $(0 + \frac{\Delta x}{2})$ and end at $(1 - \frac{\Delta x}{2})$
`sum(f(x))*dx`	The midpoint rule integral.

11.2.2. Two improvements

Newton-Cotes quadrature

A quadrature formula is an approximation of the integral of the form

$$\int_a^b f(x)dx \approx \sum_{k=0}^{M} w_k f(x_k) \tag{11.3}$$

The most widely used quadrature formulas are

$$\text{Trapeziod rule} \quad \int_{x_0}^{x_1} f(x)dx \approx \frac{h}{2}(f_0 + f_1) \tag{11.4}$$

$$\text{Simpson's rule} \quad \int_{x_0}^{x_2} f(x)dx \approx \frac{h}{3}(f_0 + 4f_1 + f_2) \tag{11.5}$$

$$\text{Simpson's 3/8 rule} \quad \int_{x_0}^{x_3} f(x)dx \approx \frac{3h}{8}(f_0 + 3f_1 + 3f_2 + f_3) \tag{11.6}$$

$$\text{Boole's rule} \quad \int_{x_0}^{x_4} f(x)dx \approx \frac{2h}{45}(7f_0 + 32f_1 + 12f_2 + 32f_3 + 7f_4) \tag{11.7}$$

Where f_i denotes $f(x_i)$ and h is the distance from one point to the next one.

Interpretation It can be shown that these rules are related to a polynomial interpolation of $f(\cdot)$ with an analytical integral. The trapezoid rule corresponds to a linear interpolation,

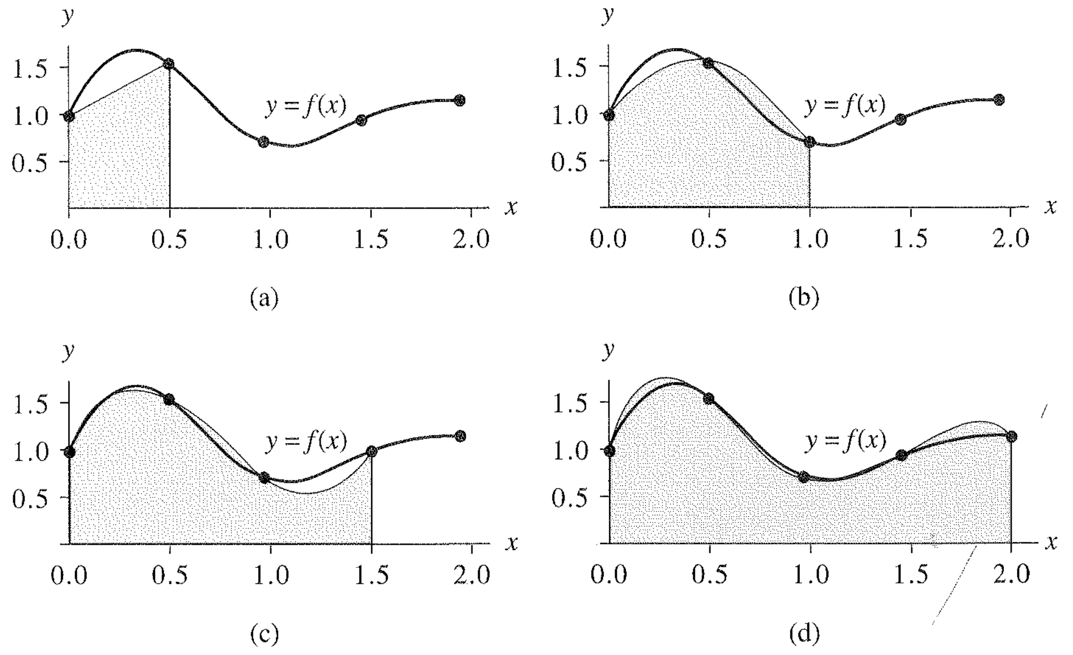


Figure 7.2 (a) The trapezoidal rule integrates $y = P_1(x)$ over $[x_0, x_1] = [0.0, 0.5]$. (b) Simpson's rule integrates $y = P_2(x)$ over $[x_0, x_1] = [0.0, 1.0]$. (c) Simpson's $\frac{3}{8}$ rule integrates $y = P_3(x)$ over $[x_0, x_3] = [0.0, 1.5]$. (d) Boole's rule integrates $y = P_4(x)$ over $[x_0, x_4] = [0.0, 2.0]$.

Figure 11.2.: [From Mathews/Fink (2004)]

Simpson's rule to a quadratic interpolation and so on. Note the in the limit, all these integration schemes will give the same result. For an illustration, see Fig. 11.2.2.

Adaptive (recursive) quadrature

Equation (11.2) tells us that the integration error of an n-th order approximation depends on the $n + 1$-th derivative. This leading term in the integration error usually varies over the interval. We can improve the precision, if the make the integration steps smaller where this derivative has a large value and wider, where the function is flatter. There are different ways to do this, but adaptive quadrature is specially appealing. The algorithm is as follows:

1. Split the integration interval in two pieces.
2. Integrate the function on each interval using two different quadrature methods. (We know that in the limit, they will all give the same result).
3. Verify on each sub-interval separately, if the target precision is achieved. If not, proceed with (1)

TIP: Have a look at MATLAB's `quad()` function by typing `edit quad`. The code is well-written and an excellent example of a recursive algorithm.

11.2.3. Infinite integral

In order to find a numerical solution for an infinite integral

$$\int_{-\infty}^{\infty} f(x)dx \tag{11.8}$$

one has to find a replacement for $-\infty$ and $+\infty$. We cannot simply choose "very large" numbers, because the choice of integration boundaries directly influences the precision, as seen in (11.2). There are three approaches to solving this problem.

Relevant region

Integrate only over "relevant" regions of $f(x)$. If $f(x) \approx 0$ for $x < a$ and $x > b$ then $\int_{-\infty}^{\infty} f(x)dx \approx \int_a^b f(x)dx$. This requires some knowledge of the function to be integrated. Very often, this is the case, e.g. we integrate over a density (it will hardy provide large contributions beyond, say $\pm 6\sigma$.

Adaptive integration

Adaptive quadrature can be used in two ways to solve this problem. Firstly, it can mitigate the effects of wide integration boundaries, as it will use a fine grid over regions of the functions that actually contribute to the integral and a coarse grid for the rest.

Secondly, adaptive integration can be used to bracket the integrand. We start with an interval $[a, b]$, which is split into strips $x_0, x_1, \ldots, x_{N-1}, x_N$. If there is a significant contribution to the integral over $[x_0, x_1]$ or $[x_{N-1}, x_N]$, then the starting interval is extended to the left viz. right. MATLAB's `quad()` implements such an approach, see example below.

Transformation of the integral

In many cases, we can transform an infinite integral into a finite one by using the inverse of a function that maps a finite interval, say, $[-1, 1]$ to $[-\infty, \infty]$. We use the change of integral formula:

$$\int_a^b f(x)dx = \int_{g^{-1}(a)}^{g^{-1}(b)} f(g(x)) \cdot g'(x)dx$$

Example: for $g(x) = \tan(x)$ we have $g^{-1}(x) = atan(x)$, which maps $[-\infty, \infty] \to [-1, 1]$.

Example 24 (Verify properties of a density.). A probability density function $f(x)$ has to fulfill two properties. (1) It is always positive: $f(x) > 0 \quad \forall x$ and (2) it is normalized to one: $\int_{-\infty}^{\infty} f(x)dx = 1$. Let us verify that the pdf of the standard normal distribution, $f(x) = 1/\sqrt{2\pi} \cdot \exp(-x^2/2)$, is a density.

`f=@(x)1/(sqrt(2*pi))*exp(-x.^2./2)`	Standard normal distribution.
`x=linspace(-100,100,200);`	Some test values for property (1); $\Delta x = 1$
`sum(f(x)<0)`	Zero, if f non negative for all test values.
`quad(f,-1,1)`	Small interval, get one s.d. percentile.
`quad(f,-1.96,1.96)`	Just verifying the 95% confidence interval.
`format long`	Need high precision display.
`quad(f,-10,10)`	Pretty close to 1 ... even better if we
`quad(f,-10,10,1E-14)`	increase the precision.
`quad(f,-1000,10000,1E-14)`	Can't fool `quad()`, even with a wide interval.
`sum(f(x))*1`	Riemann sum ($\Delta x = 1$). Not very precise.

11.3. Numeric integration of data

Given two vectors of data $x = (x_1, x_2, \dots x_n)'$ and $y = (f(x_1), f(x_2), \dots f(x_n))'$ we can approximate the integral

$$\int_{x_1}^{x_n} f(x) = \sum_{k=1}^{n-1} \frac{y_k + y_{k+1}}{2} (x_{k+1} - x_k). \tag{11.9}$$

This is just another application of the trapezoid rule. Higher-order integration methods can be considered with data, with the modification that $h = x_{k+1} - x_k$ is usually not be constant.

11.4. Transform methods

11.4.1. Introduction: Pricing a European call option

Consider the fundamental asset pricing equation

$$p_0 = E_0[M_{0,T} x_T] \tag{11.10}$$

where p_0 is the asset's price, $M_{0,T}$ the stochastic discount factor and x_T the (possibly stochastic) payoff at time T. For simplicity, we consider the risk-neutral case in which (11.10) takes the form

$$p_0 = e^{-rT} E_0^Q[x_T] \tag{11.11}$$

The payoff of a plain european call at maturity is

$$x_T = (S_T - K)^+ = (e^{s_T} - e^k)^+ \tag{11.12}$$

with the strike price K and the price of the underlying S_T and $(\dots)^+$ indicating “only the positive part of $(\dots)$”. The second notation follows Carr and Madan (1999) in using $k = \ln K$ and $s_T = \ln S_T$.

As we know, the expectation under measure Q is equal to the integral of the payoff times the (risk-neutral) probability density $q(s_T)$. We can therefore express the price of a european call $C_T(k)$ as:

$$C_T(k) = e^{-rT} E^q\left[(e^{s_T} - e^k)^+\right] = e^{-rT} \int_{-\infty}^{\infty} q(s_T)(e^{s_T} - e^k)^+ ds_T \tag{11.13}$$

Suppose it is not possible to evaluate (11.13) analytically, but that there is an analytic expression of the characteristic function of $q(s_T)$. This is the setting where transform methods shine. There are two important transform methods:

- The **Fast-Fourrier-Transform (FFT)** method pioneered by Carr and Madan (1999). This method is the de-facto standard in the finance industry and widely used for asset pricing – not only options, but also all fixed-interest products
- The novel **Cosine-Fast-Fourrier-Transform (Cos-FFT)** method by Fang and Oosterlee (2008). This method is faster, more efficient, but comparatively less known.

This section will develop our framework for transform methods using the Cos-FFT method, the next will give a short intro to the classical FFT.

Characteristic function

Before we can get started, we have to introduce a useful concept from probability theory, the characteristic function $\phi()$, which is defined as:

$$\phi(u) = E\left[e^{iux}\right] = \int_{-\infty}^{\infty} e^{iux} q(x) dx \tag{11.14}$$

The characteristic function is known in closed form for a large class of models. It has many useful properties (for more details, see Gut (2005), chapter 4).

- A characteristic function exists for all random variables
- It is symmetric: $\phi(u) = \phi(-u)$
- It is continuous.

- We can obtain the p.d.f. by integration: $f(x) = \frac{1}{2\pi}\int_{-\infty}^{\infty} e^{-iux}\phi(u)du$
- Summation. Let $S = X_1 + X_2$, then $\phi_S = \phi_{X_1} \cdot \phi_{X_2}$. This is useful as many characteristic functions used in finance are exponentially affine.

11.4.2. The Cos-FFT method

The basic idea of the Cos-FFT method is to write an expansion[1] for the density function. We start with a density $f(\theta)$ that is supported on the interval $[0, \pi]$:

$$f(\theta) = {\sum_{k=0}^{\infty}}' A_k \cos(k\theta) \qquad \text{with } A_k = \frac{2}{\pi}\int_0^{\pi} f(\theta)\cos(k\theta)\,d\theta \tag{11.15}$$

where $\sum'$ denotes a sum in which the first element (k=0) counts half.

The subsequent steps aim at making this expansion useful for a general p.d.f. with known characteristic function.

Step 1: general support. If the function $f(x)$ is supported on a general interval $[a, b]$ rather than $[0, \pi]$, we apply a simple transformation:

$$\theta = \frac{x-a}{b-a}\pi \qquad x = \frac{b-a}{\pi}\theta + a \tag{11.16}$$

Thus the COS-transform of a general function $f(x)$ reads

$$f(x) = {\sum_{k=0}^{\infty}}' A_k \cos\left(k\pi\frac{x-a}{b-a}\right) \qquad \text{with } A_k = \frac{2}{b-a}\int_a^b f(x)\cos\left(k\pi\frac{x-a}{b-a}\right)dx \tag{11.17}$$

Step 2: Relation to characteristic function. Recall (B.10) that $e^{i\phi} = \cos\phi + i\sin\phi$. This makes the expression for A_k looks almost like a characteristic function. Using a "truncated characteristic function" $\phi_1(u) = \int_a^b e^{iux}f(x)dx$, we can express A_k in terms of ϕ_1:

$$A_k = \frac{2}{b-a} Re\left\{\phi_1\left(\frac{k\pi}{b-a}\right)\cdot\exp\left(-i\frac{ka\pi}{b-a}\right)\right\} \tag{11.18}$$

Now define a quantity F_k that is based on the "normal" characteristic function $\phi(u)$:

$$F_k = \frac{2}{b-a} Re\left\{\phi\left(\frac{k\pi}{b-a}\right)\cdot\exp\left(-i\frac{ka\pi}{b-a}\right)\right\} \tag{11.19}$$

[1] In fact, it can be shown that a Chebyshev series expansion of $f(\cos^{-1}(x))$ is used. To see this better, produce a plot of the first 10 terms of (11.19) for $N = 50$.

The argument is that $F_k \approx A_k$ if $f(x) \approx 0$ outside $[a, b]$. If we choose the interval carefully, we can replace A_k by F_k, which is easily calculated from the characteristic function.

Step 3: Truncate the sum. To make the calculation numerically feasible, we have have to truncate the sum in (11.15). Our final expression for $f(x)$ is then

$$f(x) = {\sum_{k=0}^{N-1}}' F_k \cos\left(k\pi \frac{x-a}{b-a}\right) \tag{11.20}$$

Precision, problems and rules of thumb. The Cos-FFT method converges very fast (exponentially) and exhibits usually a very good behaviour. As with all FFT methods, it has difficulties with functions that have discontinuous first or second derivatives, as e.g. the chi-squared distribution. The more a p.d.f. resembles the normal distribution, the better the Cos-FFT works and the lower N is required. For the normal distribution, $N = 50$ yields excellent results. For more complicated (and less “normal-like” distributions), values of N up to 500 should be considered.

11.4.3. Recovering the density from the Cos-FFT

In a first step, we recover some p.d.f.s from the Cos-FFT and analyze the performance. This is a straight forward application of (11.20). The characteristic functions of some widely used distributions are given in the subsequent table. The results are shown in Fig. 11.3.

One Point in a	e^{iua}
Uniform on $[a, b]$	$\frac{e^{iub}-e^{iua}}{iu(b-a)}$
Triangular on $[-1, 1]$	$\left(\frac{\sin u/2}{u/2}\right)^2$
Normal with μ, σ^2	$e^{iu\mu-1/2u^2\sigma^2}$
Chi-squared with n dof	$\frac{1}{(1-2iu)^{n/2}}$

11.4.4. Option pricing with the Cos-FFT method

We discuss here only option pricing for homogeneous option pricing models (i.e. where C_t/S_t does not depend on S_t), because almost all option pricing models in the literature (Heston, Bates, Wishart, ...) fall in this class. For these models, we can separate the characteristic function into state-dependent and state-independent parts:

$$\phi(u, x) = \phi_{levy}(u)\, e^{iux} \qquad \text{with } x = \ln(S_0/K)$$

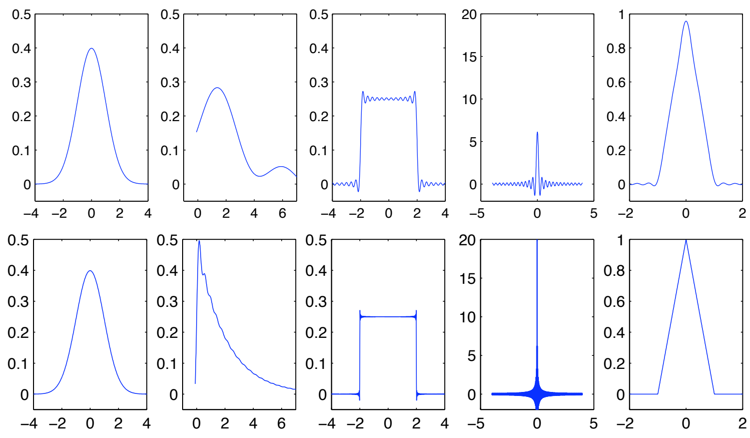


Figure 11.3.: Recovering the density using the Cos-FFT method for $N = 50$ (upper panel) and $N = 500$ (lower panel) for the normal, chi-squared, uniform, point and triangular distributions. All but the standard normal distribution have discontinuous first derivatives and are therefore subject to the Gibbs phenomenon.

We can then write the price of a Europen Call as

$$C_T = Ke^{-rT}\, Re\left\{\sum_{k=0}^{\infty}{}' \phi_{levy}\left(\frac{k\pi}{b-a}\right) U_k e^{ik\pi\frac{x-a}{b-a}}\right\} \tag{11.21}$$

with

$$U_k = \begin{cases} \frac{2}{b-a}\left(\chi_k(0,b) - \psi_k(0,b)\right) & \text{for calls} \\ \frac{2}{b-a}\left(-\chi_k(a,0) + \psi_k(a,0)\right) & \text{for puts} \end{cases} \tag{11.22}$$

and

$$\begin{aligned} \chi_k(c,d) &= \int_c^d e^y \cos\left(k\pi\frac{y-a}{b-a}\right) dy \\ &= \frac{1}{1+\left(\frac{k\pi}{b-a}\right)^2}\left[\cos\left(k\pi\frac{d-a}{b-a}\right)e^d - \cos\left(k\pi\frac{c-a}{b-a}\right)e^c + \right. \\ &\left. + \frac{k\pi}{b-a}\sin\left(k\pi\frac{d-a}{b-a}\right)e^d - \frac{k\pi}{b-a}\sin\left(k\pi\frac{c-a}{b-a}\right)e^c\right] \end{aligned} \tag{11.23}$$

$$\begin{aligned} \psi_k(c,d) &= \int_c^d \cos\left(k\pi\frac{y-a}{b-a}\right) dy \\ &= \begin{cases} \frac{b-a}{k\pi}\left[\sin\left(k\pi\frac{d-a}{b-a}\right) - \sin\left(k\pi\frac{c-a}{b-a}\right)\right] & k \neq 0 \\ d-c & k = 0 \end{cases} \end{aligned} \tag{11.24}$$

where $y = \ln(S_T/K)$.

NOTE: Not all characteristic functions in the literature are expressed in a way that is compatible with the cosine-FFT method. For example the 2 factor Heston model (Christoffersen, Heston and Jacobs 2009), will not work as expressed in the paper, because the characteristic function as stated in the paper diverges for zero[2]. By rearranging the terms, on can see that in the limit $\phi \to 0$, the characteristic function converges towards one. It is safe to replace the first evaluation of the characteristic function with one for all applications of the Cosine-FFT method.

11.5. The FFT method for option pricing

11.5.1. Introduction

The Fast Fourier Transform (FFT) has long been known in signal processing and electrical engineering. It is a method to approximate the Integral

$$X(v) = \int_{-\infty}^{\infty} e^{-ivk} x(k) dk \tag{11.25}$$

with a sum of the type

$$X(k) = \sum_{n=1}^{N} x(n) e^{-\frac{2\pi i}{N}(k-1)(n-1)} \qquad 1 \leq k \leq N \tag{11.26}$$

It has been introduced to finance in paper by Carr and Madan (1999). The main advantage of the FFT method is the fact that we can calculate sums (which stand for integrals) very fast and for many different values of k in parallel.

Again, we start from (11.13). We find the inverse Fourrier Transform (iFT) and use the FFT to calculate numerical values for $C_T(k)$. This is exactly the FFT method for option pricing. Its elegance lies in the fact that the first step – finding the iFT – is simpler than it seems. The iFT is an integral of C_T or a double integral of q_T. However, it can be transformed into a simple integral of the characteristic function of q_T.

Modified call price

Before we are able to perform the two steps as described above, we have to overcome one obstacle. For a zero strike price ($K \to 0; \quad k \to -\infty$) the call price $C_T(k)$ becomes constant and therefore the integral (11.25) will diverge in this point.

The solution is to introduce a dampening factor $e^{\alpha k}$, which goes to zero as $k \to -\infty$. The need for α can also be seen from the resulting (11.30), which would diverge for $v = 0$

[2]A characteristic function evaluated at $X = 0$ is always one, as one can easily see by inspecting (11.14).

in the absence of α. We define a modified call price c_T:

$$c_T(k) = e^{\alpha k} C_T(k) \tag{11.27}$$

There is no interpretation for c_T, it just ensures convergence of the integral (11.25). Once we have calculated c_T, we invert (11.27) to get $C_T = e^{-\alpha k} c_T(k)$. Thus we can give the full procedure for FFT option pricing:

1. Multiply C_T (eq. 11.13), for which the iFT would not converge, with the damping factor $e^{\alpha k}$; obtain the modified call price c_T
2. Analytically find the closed-form solution for the iFT of c_T by using the characteristic function of $q(s_T)$
3. Apply FFT to obtain numerical values for $c_T(k)$
4. Multiply $c_T(k)$ with $e^{-\alpha k}$ to obtain $C_T(k)$

The full procedure for FFT option pricing

$$C_T(k) = \underbrace{\underbrace{e^{-\alpha k}}_{c_T \to C_T}\ \underbrace{\frac{1}{\pi}\int_0^\infty dv\, e^{-ivk}}_{FFT}}_{numerically}\ \underbrace{\int_{-\infty}^\infty dk\, e^{ivk}\ \underbrace{e^{\alpha k}}_{C_T \to c_T}\ \underbrace{e^{-rT}\int_{-\infty}^\infty ds_T q(s_T)(e^{s_T} - e^k)^+}_{C_T(k)}}_{iFT=\psi_T(v),\, analytically} \tag{11.28}$$

11.5.2. Analytical part: steps 1-2

Inverse Fourier transformation

Let us now calculate the inverse Fourier transform of the modified call price. We evaluate:

$$\begin{aligned}
\psi_t(v) &= e^{-rT}\int_{-\infty}^\infty e^{ivk} c_T(k) dk \\
&= e^{-rT}\int_{-\infty}^\infty e^{ivk}\int_{-\infty}^\infty e^{\alpha k}(e^s - e^k)^+ q(s) ds\, dk \\
&= e^{-rT}\int_{-\infty}^\infty ds\, q(s)\int_{-\infty}^\infty dk\, e^{ivk} e^{\alpha k}(e^s - e^k)^+ \\
&= e^{-rT}\int_{-\infty}^\infty ds\, q(s)\int_{-\infty}^s dk\, e^{ivk} e^{\alpha k}(e^s - e^k) \\
&= e^{-rT}\int_{-\infty}^\infty ds\, q(s)\int_{-\infty}^s dk\, \left(e^{s+k(\alpha+iv)} - e^{k(1+\alpha+iv)}\right)
\end{aligned}$$

$$
\begin{aligned}
&= e^{-rT}\int_{-\infty}^{\infty} ds\, q(s)\left[\frac{e^{s+k(\alpha+iv)}}{\alpha+iv}-\frac{e^{k(1+\alpha+iv)}}{1+\alpha+iv}\right]_{-\infty}^{s} \\
&= e^{-rT}\int_{-\infty}^{\infty} ds\, q(s)\left[\frac{e^{s(1+\alpha+iv)}}{\alpha+iv}-\frac{e^{s(1+\alpha+iv)}}{1+\alpha+iv}\right] \\
&= e^{-rT}\int_{-\infty}^{\infty} ds\, q(s)\frac{\left[(1+\alpha+iv)-(\alpha+iv)\right]e^{s(1+\alpha+iv)}}{(\alpha+iv)(1+\alpha+iv)} \\
&= \frac{e^{-rT}}{\alpha^2+\alpha-v^2+i(2\alpha+1)v}\int_{-\infty}^{\infty} ds\, q(s) e^{s(1+\alpha+iv)} \qquad (11.29) \\
&= \frac{e^{-rT}}{\alpha^2+\alpha-v^2+i(2\alpha+1)v}\,\phi\big(v-(1+\alpha)i\big) \qquad (11.30)
\end{aligned}
$$

Where we used the definition of the characteristic function (11.14) for the last step. Note that the result would diverge for $v \to 0$ in the absence of α.

Note: The log strike price k is no more an argument of ψ. This is achieved by exchanging the integrals.

Choice of α

It is clear that one condition is $\alpha > 0$, because otherwise the damping factor would not dampen for $k \to -\infty$. Carr and Madan (1999) use a value of 1.2 for α and give the criterion that

$$E[S_T^{\alpha+1}] < \infty \qquad (11.31)$$

Note that (11.29) *is* an expectation! If α is too low, it will not dampen C_T sufficiently, yielding wrong results for small values of k; if it is too large, the results for high values of k will suffer. To verify the choice of α, I suggest two checks:

1. Perform a stability analysis. Compare the results for a range of values of α. If the results strongly depend on α even within a small range, be cautious.
2. Check the known limits for the call prices: $C_T \to (S_0 - K)$ for $K \to 0$ and $C_T \to 0$ for $K \to \infty$. The trick is to verify these limits at the right values of k. They will be violated for *extremely* low and high values of k, they should however hold for *moderately* low and high values of k, e.g. for a moneyness of 0.5 and 2.

11.5.3. Numerical part: steps 3-4

To transfer $\psi_T(v)$ back to C_T, apply the FFT and correct for the dampening factor:

$$C_T(k) = \frac{e^{-\alpha k}}{2\pi} \int_{-\infty}^{\infty} e^{-ivk} \psi(v) dv = \frac{e^{-\alpha k}}{\pi} \int_0^{\infty} e^{-ivk} \psi(v) dv \tag{11.32}$$

The second equality holds because of a symmetry argument: An infinite integral over a symmetric $\times$ antisymmetric function is zero, over a symmetric $\times$ symmetric function is nonzero. c_T is real (because C_T is real) and the real component of e^{-ivk} is symmetric, thus the real component of $\psi(v)$ must be symmetric, as well. This implies $\int_{-\infty}^{\infty} \dots = 2\int_0^{\infty} \dots$.

An infinite integral cannot be approximated using a sum. Therefore, we limit the upper integration boundary in (11.32) to a. The choice of a will be discussed later. If we slice the interval $[0, a]$ into N pieces, the integral (11.32) can be approximated as

$$C_T(k) \geq \frac{e^{-\alpha k}}{\pi} \int_0^{a} e^{-ivk} \psi(v) dv \approx \frac{e^{-\alpha k}}{\pi} \sum_{j=1}^{N} e^{-iv_j k} \psi_T(v_j) \frac{a}{N} \tag{11.33}$$

with $v_j = \frac{a}{N}(j-1)$. Carr and Madan use the symbol $\eta = \frac{a}{N}$ for the integration step.

The FFT can do more than approximate the integral (11.32) with the sum (11.33); it can evaluate (11.33) simultaneously for N equally spaced values of k. Carr and Madan (1999) propose a range of log strike levels from $-b$ to b, which implies a spacing of $\lambda = \frac{2b}{N}$. The idea is that this includes at the money calles, for which K is a reasonably low number and thus $k \approx 0$. This would fit perfectly if we normalized S_0 to 1, in which case K would be approximately the moneyness.

The solution of the FFT becomes a N-dimesional vector, where the elements are indexed by the argument $k_u = -b + \frac{2b}{N}(u-1)$. Thus the result vector can be written as:

$$\mathbf{C_T}(k_u) \approx \frac{e^{-\alpha k_u}}{\pi} \sum_{j=1}^{N} e^{-iv_j k_u} \psi_T(v_j) \frac{a}{N} \qquad \text{for } u = 1, \dots, N \tag{11.34}$$

Equation (11.34) looks pretty much like (11.33). The difference is that (11.34) is a vector; remember that we sum over j, while u is the index of the vector of results. Finally, we insert the expressions for k_u and v_j to get:

$$C_T(k_u) \approx \frac{e^{-\alpha k_u}}{\pi} \sum_{j=1}^{N} e^{-i\frac{2\pi}{N}(j-1)(u-1)} \underbrace{e^{ibv_j} \psi_T(v_j) \underbrace{\frac{a}{N}}_{\eta}}_{X(j)} \qquad \text{for } u = 1, \dots, N \tag{11.35}$$

MATLAB example: `myfft_optionprice.m`

11.6. PC Lab: Comparing FFT and COS methods

See `myfft_optionprice.m`

11.7. Exercises

To follow

12. How to write a successful program

References: Kerrighan and Plauger; Kuniciky chapter 10; MATLAB guide "Program Development" (Version 7), sections 1, 3, 4, 11, 12

12.1. Introduction

Most programming projects start small and are not intended to grow. This is, however, exactly what happens: you add one aspect, then another one and so on. After some time, you are confronted with a large, complex program and nobody – including yourself – understands how it really works. This will almost surely cause you a lot of pain and cost a lot of time. Maintaining a good programming style may take some extra work, but this investment will almost surely pay off. There are several reasons to use good style:

Errors. Good style helps avoid errors. This saves time (to search for errors) and embarrassment (if you don't find them).

Memories. Half a year after finishing a project, most of us have forgotten the details and will have difficulties in modifying/amending a badly-written program.

Collaboration. You might start a work/research collaboration. It may also happen that someone else has to maintain/use your program after your have changed jobs.

Publications. The process of publishing a paper is often a back and forth between author and referee, where each iteration may easily take several months. You may be asked to perform an additional estimation/robustness check months or even years after send a paper to publication. Scientific also journals ask increasingly for details about the data and estimation procedure, to render every aspect of a publication transparent.

12.2. Elements of good programming style

In 1974, Kerrighan and Plauger published a book called "The elements of programming style". Remarkably, most of this book is valid today. The following guidelines have been inspired by their 60 rules and include some personal observations of the author.

12.2.1. Planning

Start with an empty sheet of paper. It is very important to separate planning from coding. Do not code right away, make a plan first. Do not think too much about technical details in this phase. Write down your ideas in "pseudo code", like *load the data; calculate the regression coefficients; check the error terms.*

***Make a flowchart.** A graphical representation makes it easy to discover dependencies and the correct sequence of commands.

Modularize. Code that is used more than once should be put into a function to avoid inconsistencies (you make a change at the first occurrence but not in the second one.) Beyond this, functions have two advantages. They can be easily re-used in other projects and they hide code. Putting boring stuff like data input into a function will improve the readability of the code. Modules also facilitate testing and debugging. This also means that each function should do one thing well (and no more).

Solve a sufficiently general problem. Your requirements will probably increase with time. If possible, solve for the n-dimensional case (instead of hard-coding 2 dimensions).

Think about the data format. This is important for several reasons: a badly designed data format makes coding more difficult and the program less readable. When designing the data format, keep in mind that estimations rarely produce perfect results at the first run. You will probably need tens, maybe even hundreds of runs with different subsets of the data/model or starting values. You will want to save the *results* and all *envelope information*, e.g. which exact settings (model restrictions, data set) were used. On the other hand, you will not want to save all the *input data*. Consider using one data structure for each of the above mentioned data categories.

Use standards for your data format. When using a matrix, store individual observations in rows and variables in columns. Think about possible "holes" in the data (e.g. different holidays in different countries) and how to handle them.

Convert immediately after input. Your data may not come in the correct units (e.g. daily vs. yearly returns). To avoid inconsistencies, convert the data immediately after input. Converting in an external program (e.g. a spreadsheet) is not a good idea, because it increases complexity and constitutes a separate source of errors.

12.2.2. Coding

TIP: It's a good idea to write code with a reader on your mind: your supervisor, a colleague or your grandma. Would they understand what you are writing?

1. This naming convention applies to variables, functions and MATLAB save files.
2. All names are self-explaining English words, usually not abbreviated and at least 5 letters long. I use the singular even if numerous entries are expected. Example: `price, duration, strike`
3. Names of variables start with a lowercase letter, names of functions with an uppercase.
4. Exceptions to (1) and (2) are widely used single-letter symbols and widely used acronyms. For both, I use only uppercase letters. Examples: `X,Y` in regression analysis or `FFT, GDP`.
5. Uppercase letters concatenate words. The following rules apply to concatenated names:
   - Names of *variables* start with the most general term and end with the most specific (the latter usually denoting a certain method). Examples: `optionPriceFFT, optionPriceAnalytic, optionPriceData, optionStrike`
   - Names of *functions* start with the name of the project, followed by the object of the function, what it does and finally the specific method. Examples: `WishartOptionPricingFFT, WishartOptionPricingCosfft, WishartOptionPlotting`
6. To avoid confusion, I use the prefix "`my`". This may be necessary if one variable is used locally to calculate a variant of a model or if the name conflicts with a MATLAB command. Example: `myOptionPrices, myDate`

Figure 12.1.: My personal naming convention

Write clearly – don't be too clever. Most of the advanced logic in code makes it hard to read. Try to solve problems with simple solutions, perhaps a few lines longer and not as efficient – but readable. Optimize for speed and memory only where necessary. In short: Let the machine do the dirty work.

Avoid reparametrizations. Whenever possible, variables should contain a value that is readily available. This means that interest rates should be expressed in fractions to avoid having to divide by 100. This also means that loop variables should be readily applicable, e.g. that `t` means "today".

Use a standard code layout. This includes a standard program structure (see p. 28) and a standard structure for functions (p. 30).

Adopt a naming convention that ensures that your names mean something, both for variables and functions. Self-explaining names make a program easier to read and understand. Also, choose names that cannot be easily confused. A naming convention will always be subjective. An example for a naming convention (my personal one), can be found in Fig. 12.1.

Adopt a convention for vectors and matrices. One of the most common MATLAB errors is a wrong dimension in a matrix multiplication. This can be avoided, if all vectors are stored as column vectors.

Avoid temporary variables and do not use one variable for different purposes within a program. Both are a sources of confusion.

Produce readable code. The following things make code much more readable: Indent at every `if`, `case`, `for`, `while` and outdent at every `end`. Align the assignments of variables and use cells as well as empty lines to structure the code for the reader. Break long equations using three dots at a logically suitable place. Use parenthesis so that the code is readable and use spaces inside parenthesis.

All constants in the first block. Take the program on page 74 as an example: we ran it with $n = 100$ and $n = 1000$. It would be very difficult to change the value, if n were defined at some place in the program. Every constant that *might* be changed later should be put in the first few lines, ahead of the first command. Also, avoid any user input in your programs.

Use library functions. Do not re-invent the wheel, because this is time-consuming and a potential source of errors. If there is a standard function that solves a part of your problem, use it. Alternatively, build your own library, see section D.5

Write and test a big program in small pieces. An incremental/evolutionary approach helps control complexity. Testing a program in parts is especially useful, because finding an error in a large code is very difficult.

Use versioning. Every time you have achieved a mile stone, freeze the code and start a new version. In the case of errors, you can roll back to the last stable version.

***Don't compare floating point numbers for equality.** Remember exercise 3.2. MATLAB said that $(A^{-1})^{-1} \neq A$, because of rounding errors. Instead of comparing for equality, verify if some metric of the difference is below a threshold, see page 38.

***Take care to branch the right way on equality.** Is the condition "less" or rather "less or equal"?

12.2.3. Documenting

Code commenting

Make every comment count. A comment should explain the logic of the program (what *you* have been thinking). Do not explain the MATLAB commands line-by-line – this is better done by the MATLAB online help.

Use standard documentation for programs and functions. This automatically creates a help text for your function. Include name, inputs, outputs, usage and references.

Refer to the literature. Make a clear reference to your source when implementing published algorithms. This can go as far as mentioning equation numbers from a paper at the end of each line.

Choose an appropriate format for the comments. Comments at the beginning of a cell should run over whole lines. Use comments at the end of the line to explain naming, especially when creating variables.

Document all changes. Place a changelog at the end of the preamble.

Make sure comments and code agree. Update your comments when you update your code. Comments should be above or on the right side of the explained command.

User Documentation

Just as comments in the code, documentation is indispensable for larger projects. Good documentation need not be voluminous, it should rather concentrate on the main issues. Try to avoid overlap with the code comments.

Describe the program. Write in few plain English sentences what the program does. Give some few references for the details. (In most cases, we use methods that are published somewhere. Reference these papers.

User guide. How to run the program (best add a demo program and demo data with no input and no preparation needed).

Customization. Explain the main parameters (i.e. things that the user can change and that influence the result). State how and where (in the code) they can be changed.

Document your data format. Describe in detail the structure of input and output files, including the units. A common source of errors are percentage values: 5.5% can be stored as 5.5 or 0.055 – we recommend the latter. Document any necessary data conversion steps outside the program, e.g. in a spread sheet.

Add a readme or contents.m file. As Matlab projects often contain a lot of files for functions, it is not easy to identify which is the main program. Add a file called readme.txt in which you explain how to run the code; alternatively create a file called contents.m which highlights the main program and explains how all functions work.

12.2.4. Testing

Test in an organized, modular way. Make a test plan. The best thing is to devise a test suite. First, check every function separately to reduce complexity. Use tools like the Desktop/Profiler to see if all the code ran during the test.

Clearly mark tested code. Write the test date in the code. Together with the changelog you can see whether the current version is tested or not.

Test at least twice. Some programs only fail at the second trial. In the case of a simulation, verify if the result is (slightly) different at every run.

Test for different input. A program may just by chance work for a certain parameter set. Test different ones. Where applicable, test for deliberately false input.

***Test programs on their boundary values.** A special case of the above rule. Some errors can only be detected at boundary values, e.g. for an interest rate of 0%.

Check some answers by hand. If possible, perform some calculations (for a simple case) by hand. If this is not possible, check results for plausibility.

Do not only look for errors; take warnings seriously. Some MATLAB warnings are just annoying, still check if there is a problem behind them.

Include integrity checks in your code. It is a good idea to check intermediate results for plausibility. If – during an optimization – you encounter an interest rate of −500%, then there is probably an error. Issue a warning in this case.

12.3. Finding code and quoting it

A lot of code can be found on the Internet or in books and papers (see section 2.5). Using this code is certainly better than re-inventing the wheel. There are three things to keep in mind when using foreign code:

1. The code may be wrong or may work only in some special cases. We don' t know the application that the author had in mind when writing the code.
2. Code is not “innocent”. It is a formalized theory. Thus be sure that not only the code is correct, but the underlying theory is correct and applicable to your case.
3. Do not forget to quote according to copyright and academic standards.

Rules for correctly quoting code

- **Whole function** Normally there is a © in the header. If not, add: `Code by ...`
- **Program idea** In the introductory comment, add "based on a program by ..."
- **Few lines of code** Clearly mark the start and ending of the foreign code.

```
% ---- Code by NN ---------
The few lines ...
% ---- End code by NN ---------
```

12.4. Incomplete list of frequent MATLAB errors

Missing brackets. See p. 27.

Misplaced brackets. Consider the sum of squared errors $SSE = \sum_i (y_i - x_i\beta)^2$. It is wrong to write `sum(y-x*beta)^2`. Correct: `sum((y-x*beta).^2)`.

Manually calculated values. Like `a=1.67` instead of `a=5/3` or `x^0.33` instead of `x^(1/3)`.

Mixing row and column vectors. One of the most abundant error messages in MATLAB is `??? Error using ==> ... Matrix dimensions must agree`. An example is to add a row and a column vector. A simple remedy is to add a "transpose" (i.e. a dot-apostrophe `.'`). Much better is to think about the general data structure and to clean up the whole program with respect to this issue.

Equality with real numbers. See p. 38

Logical conditions with matrices. See p. 39. Depending on problem use `isequal()`, `any()`, `all()`.

Typo in name of variable. MATLAB names are case sensitive. A slightly mis-spelled name makes MATLAB create a new variable. See PC lab.

Improper initialization. Best start every program with `clear` to avoid confusion with existing variables.

Directly assigning elements of a vector/matrix. Be careful when directly assigning an element of a vector or a matrix. See PC lab for example.

Indiscriminate load. Imagine you saved your whole workspace in the file `mystuff.mat`. If you execute a `load mystuff` in the middle of a program, there is a good chance you will also change a few loop variables or other items. Always use

```
load <filename> <variable(s)>.
```

12.5. Dos and don'ts in MATLAB

Don't use break and continue. They are usually signs of a bad logic of the program.

Don't use tick2ret. You don't need a toolbox command for calculating returns.

Use for and while loops appropriately. With a few tricks, one can exchange for-loops and while-loops. Although this does not produce errors, it usually lengthens the code and perverts the logic of a program. Therefore it should be avoided.

```
for k = 1:10
    disp(k)
end
```

Good: For loop as intended

```
k=0;
while k ≤ 10
    k = k+1;
    disp(k)
end
```

Bad: While loop replaces for loop

```
u=1; k=0;
while u > 1E-4
    k = k+1;
    u = u/2;
end
```

Good: While as intended

```
u=1;
for k = 1: 10000000
    u = u/2;
    if u ≤ 1E-4
        break
    end
end
```

Bad: For loop replaces while using `break`

12.6. PC lab: How good style prevents errors

Typo in the name of a variable

`sum=1;`	Initialize variable
`Sum=sum+2;`	Running sum ...
`sum=sum+3`	We would expect 6, but we only obtain 4

Directly assigning elements of a vector/matrix

Imagine you run a simulation of stock market returns month by month.

```
nDays=[31 28 31 30 31 30 31 31 30 31 30 31];
for m=1:12                  Run simulation month by month ...
  days=nDays(m);            Number of days in this month
  for d=1:days;             Loop over number of days in month
    ret(d)=0.02*randn;      Fixed volatility for simplicity.
  end                       End of daily loop.
  subplot(6,2,m);           Select subplot.
  plot(1:days,ret)          Here the error occurs.
end
```

12.7. Exercises

Exercise 12.1. Find suitable names for functions that do the following:
(*a*) Calculate the implied volatility of one option contract via bisection.
(*b*) Calculate the price of an option in the Heston model using the Cosine FFT.
(*c*) Create a plot of the implied volatility surface.
(*d*) Calculate all prices of an option panel in the Heston model using Monte Carlo simulation. (*e*) Produce a file containing the results of your estimation as a LaTeX table
(*f*) Produce a file containing the results of your estimation as a plain text table

Exercise 12.2. Each of the following should be half a page to maximal one page:
(*a*) Create your personal naming convention. (*b*) Create a personal checklist for coding conventions and coding style.Find a way to organize your checklist. (*c*) Write down your personal testing procedure.

Exercise 12.3. Search the web for functions that can do the following. Start your search at the list given in Section 2.5, but find at least on function on a webpage not listed in Section 2.5. (*a*) Calculation of Newy-West errors. (*b*) Calculation/plot of the kernel smoothing regression. (*c*) Estimation of a GARCH model. (*d*) Calculation of the price of an Asian option in the Heston model using Monte Carlo simulation.

13. Improving a program

13.1. Debugging

Debugging an error can be a tedious and frustrating task. The best way to avoid such a situation is to adhere to the rules of good programming style. There will still be occasions in which one has to find an error. There are two types of errors. Errors with error messages are mostly syntax errors, occasionally they are the sign of a deeper routed problem. Errors with wrong results are almost surely logical errors. These are much harder to debug.

Errors with error messages

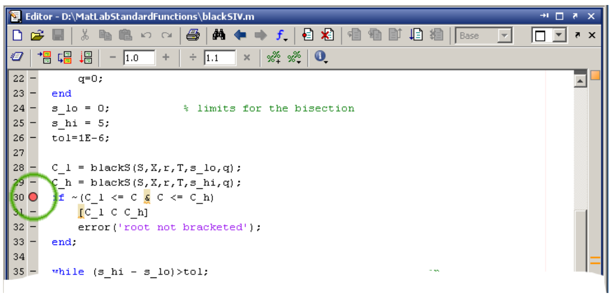
- Read the error message. Read every word in it.
- Note that error messages are *sometimes* wrong.
- Go to the line number indicated. The error is either in this line or in any line *above*.
- If you do not find the error, use the techniques from the subsequent section.

Errors with wrong results

- Check input and parameters. (Computer law: GIGO = garbage in, garbage out.) Maybe the code is correct, but there is a problem with the data – mostly a conversion problem.
- The error may have happened early in the program.
- Perform a `clear`. If the error is gone, this is *not* the solution. It is, however, a hint where to look for: initialization.
- Track intermediate results using `echo, disp()` or by removing semicolons.
- Better, use the debugger and breakpoints to see what is going on.
- Simplify the problem by commenting out some lines of code.
- Go back to an earlier, working version.
- Get some distance and think through the problem.

The MATLAB debugger

To start debugging, simply insert a break-point in the program editor by clicking to the right of the line number (see Fig. 13.1). When executing the program, MATLAB will



```
22 -        q=0;
23 -    end
24 -    s_lo = 0;           % limits for the bisection
25 -    s_hi = 5;
26 -    tol=1E-6;
27
28 -    C_l = blackS(S,X,r,T,s_lo,q);
29 -    C_h = blackS(S,X,r,T,s_hi,q);
30 ●    if ~(C_l <= C & C <= C_h)
31 -        [C_l C C_h]
32 -        error('root not bracketed');
33 -    end;
34
35 -    while (s_hi - s_lo)>tol;
```

Figure 13.1.: The MATLAB debugger: just click in the dash next to the line number – a red dot (here in line 30) indicates a break point. Right-click to obtain a conditional break point.

stop before executing this line. The command prompt changes to `K>>`. Have a look at the workspace to see intermediate results. You can even produce plots or perform diagnostic calculations in the command window. To continue, use one of the special debugger commands.

Debugger commands

`dbstep`	Move to the next executable program line.
`dbstep` *n*	Move *n* lines forward.
`dbcont`	Continue until you encounter the next break point.
`dbquit`	Exit the debug mode and the program.
`dbclear all`	Remove all break points

NOTE: When stopping at a breakpoint inside a function, MATLAB will show the workspace of the current function (which is normally not visible and discarded at exit).

TIP: Other useful commands for debugging: `echo()`, `whos()`, `size()` as well as `lasterr()`, `lasterror()`, `lastwarn()`. Commands you can use to help others debug their code: `warning()`, `error()`

Removing errors

- Change one thing at a time and roll back changes that have no effect.
- When performing a change, think of similar code lines (maybe you have copied, pasted and slightly altered this line).
- Versioning is especially important when debugging.
- After removing an error, start testing from scratch.

TIP: The MATLAB Programming Tips manual contains a valuable section on debugging. This manual comes digitally with every copy of MATLAB and is available on the Mathworks homepage.

13.2. Increasing a program's efficiency

If two programs take the same input and produce the same output, which one is better? There are three criteria to decide this: Speed, memory requirements and elegance. As discussed in the previous section, an elegant program follows the rules of good coding and achieves the required task in an as simple as possible way. Elegance is always a desirable feature and it is worth the effort to improve a program along this dimension, e.g. by shortening a program (13.3).

As for the execution speed, we should be aware that writing a program is an optimization problem. Execution speed can be reduced, but usually at a large cost to your time. Speeding up a complex Monte Carlo simulation from 1 month execution time to 1 day sounds good, but if it takes you 2 months to improve the code, you have actually lost time. With today's performant and cheap computers, your time usually the most expensive resource. Under normal conditions we will therefore not spend a lot of time optimizing a program for speed or memory usage. Only if a program is excessively slow or if it crashes due to a memory overflow, we have no choice but to intervene.

There are three approaches to both problems: reduce the requirements, use single precision or improve the code. However, the main principle of optimizing code is:

Don't fix it, if it ain't broken.

13.2.1. Speeding up a program

Measuring the execution time

Test the speed of a program at least twice. The first run is often slower, because MATLAB has to load the libraries. To time a program, use the `tic` and `toc` commands.

Example: timing a piece of code

`tic;`	Start the clock.
`<some lines of code>`	
`toc`	Stop the clock and display the time since `tic`.
`t=toc/N`	We can even do calculations with `toc`.

NOTE: Very often, we want to know how long a program is going to run beforehand. In such a case we run a sub-sample of the calculations (e.g. reduced number of observations, reduced number of Monte Carlo iterations) and estimate the total running time. Such an estimate is usually reliable, unless an adaptive algorithm is involved, i.e. the execution time depends on the data. Examples for such cases are optimisations and adaptive quadrature algorithms.

Improve the code

- First, use the profiler to identify bottlenecks. In most cases, just a few lines (often within a loop) take up 80% of the execution speed. Focus on these lines.
- Check for unnecessary multiple executions of code in loops. Examples: data conversions (could be done before the loop) and statistics (could be calculated after the loop).
- Consider using an alternative algorithm, e.g. a better optimizer or CosFFT instead of FFT.
- Vectorize or parallelize your algorithm, see below.

Reduce the requirements

To be on the safe side, we perform calculations quite often with more precision than needed. To speed up a program, consider the following:

- Restrict your sample, e.g. to only one year or by choosing a sub-sample
- Reduce number of Monte Carlo draws.
- Lower the precision in optimization and integration.
- Use an approximate solution (e.g. a linear approximation).

Use single precision

Linear algebra functions (e.g. matrix multiplication, matrix inversion, eigenvalues) execute up to twice as fast in single precision. Furthermore, single precision variables require half the memory. A comparison of single and double precision is given in table 13.1.

MATLAB-specific speed tricks

- Try to vectorize a loop (see example below). Vectorizing, however, may increase the memory requirements.
- Built-in functions are usually faster than self-written ones.
- Consider re-writing the bottleneck (including the loop) as MEX-function in C.
- For-loops can be accelerated if all arrays are pre-allocated. For more details, read the documentation for the "MATLAB JIT/Accelerator".

Example 25. There are several ways how to sum all numbers up to N. Their execution time varies by a factor of up to 100, as can be seen in Fig. 13.2. The code is shown in the following table (see `manyways.m`):

Loop	`total=0; for k=1:N; total=total+k; end`
Vectors	`vec1=1:N; vec2=ones(N,1); total=vec1*vec2;`
MATLAB	`vec1=1:N; total=sum(vec1);`
Gauss' formula	`total=N*(N+1)/2;`

13.2.2. Reducing memory usage

Reducing the requirements not only increases speed but also reduces the memory usage. Beyond this, most of the following measures for reducing memory usage tend to slow down a program. These are:

- Slice the problem into independent tasks and discard intermediate results as soon as possible. Use `clear <variable_name>`. Keep only values that are used later on (e.g. in a time-series Monte Carlo, keep only the last day and discard the path).
- Functions automatically clear all internal variables once they are ended. Moving large amounts of data from and to functions is however slow and memory intensive. Functions can help reduce memory usage for calculations that have only a few input and output parameters but need large variables for the in-between calculations.
- If you cannot discard a large intermediate result, use `save` to write it to disc, then `clear` it and `load` it later when needed. (This is very slow).
- De-vectorize. Vectorizing is fast in MATLAB, but it requires a lot more memory. The memory requirement increases by the size of the vector, i.e. manyfold.

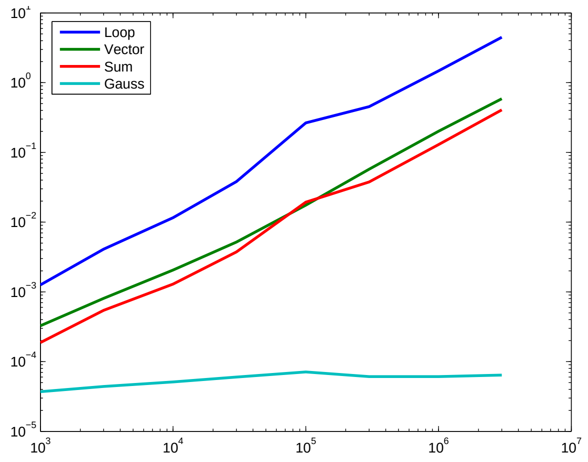


Figure 13.2.: Execution time in seconds for different algorithms to calculate the sum of all numbers up to N. Output produced by `manyways.m`

type	memory	eps	realmax	rule of thumb
single	4 bytes	`1.192e-07`	`3.4e+38`	7 significant digits
double	8 bytes	`2.22e-16`	`1.8e+308`	15 significant digits

Table 13.1.: Comparison of single and double precision variables

Example: Inefficient Monte Carlo simulation

```
T=50; N=100; K=1;                     Setup. T time steps, N draws, K = strike.
paths=exp(cumsum(0.05*randn(T,N)));   Geometric Brownian motion in one line.
val=paths(end,:)-K;                   Step 1 for option payoff
option=mean(val .* (val>0))           Step 2 for option payoff
whos paths                            The variable paths uses 40kB
```

Example 2: Efficient Monte Carlo simulation

```
for k=1:N                             Assume T, N, K from previous example.
  path1=exp(cumsum(0.05*randn(T,1)));Geometric Brownian motion in one line.
  pathEnd(k)=path1(end)-K;            Step 1 for option payoff
end
option=mean(pathEnd .* (pathEnd>0))   Step 2 for option payoff
whos path*                            The variables path1 and pathEnd use 1.2kB
```

- Single-precision numbers require half the memory of double-precision numbers:

```
a=1:1000;          A vector with 1000 elements
b=single(a)        The same vector, stored as single precision variable.
c=int16(a)         The same vector, stored as integer variable.
whos               The third column (Bytes) shows the memory used.
```

13.3. Dealing with complexity

Many research projects encounter a breakdown point, when complexity simply "grows over your head". Complexity increases the probability of errors exponentially and leads to the possibility of hitting a "brick wall" in the development process. This section deals with reducing and managing the unavoidalbe complexity in every research project.

Complexity has many faces, but one major source of complexity in research is the fact that there are so many free parameters. Quite often it is our urge to try every possible combination of model × dataset × metric × estimation method. We may want to estimate the baseline, full, extended, restricted, linearized or simplified model. We may want to do this in and out of sample, for different frequencies, with or without data cleaning, for different horizons, timespans, firm characteristics .... We finally may want to calculate relative or absolute errors, different weightings, different econometric methods. And possibly everything combined with anything.

Avoiding complexity saves a lot of pain. The best rule for doing so is: Be an economist, not a data snooper. Define clear research question, and dont't just play around with the data. Survey the literature for standard datasets (often US), standard estimation procedures and standard baseline models. Making a contribution to research means deviating from one or two of these standards, but not from all of them! A new model, for example, is much more appreciated by the reader if it is applied to a know dataset and compared to a known model.

Define an econometric (i.e. measurable) criterion for what is "best" in the remaining choices *before* starting to code. This avoids the temptation of data snooping and accellerates the selection of the remaining parameters of your project. Such a criterion also makes it possible to let the code identify the "best" solution, saving you the complex task of evaluating all solutions by hand.

Planning for complexity pays off. Almost every project grows more complex than anticipated, so code for complexity right from the start. It is especially useful to ...

- Plan data the format and devise a naming convention
- Plan a flexible code structure (functions) based on the assumption that key project parameters are likely to change
- Not to try all combinations of parameters/data/settings.
- Create a research workflow in separate programs or functions:
  Data preparation $\longrightarrow$ model estimation $\longrightarrow$ evalutation $\longrightarrow$ reporting

Changes increase the complexity, so a good management of changes avoids a lot of headache. When a change is necessary, follow through with it before doing anything else. Changes usually have several implications throughout the code. Think of all parts of the code/functions that need to be adapted. When changing the data format, change all existing files at once. Write a converter program to automate this step. Try to avoid conflicting code/formats/naming conventions when making changes.

Rewriting code may be a good investment. One way to reduce complexity is rewriting your code. This may be a good idea once you have overcome the main difficulties at the beginning of a project. It may also help to adhere to the standards of good programming (if the first version of your program did not, possibly due to perceived time constraints). Rename variables/functions according to your (updated) naming convention. Re-write and shorten your code using more functions (see also section 13.4).

Evaluate your data structure. After several project expansions, the data structure is often not up to the task anymore. This is usually compensated by creating numerous variables. Changing the data format is tricky and may require a lot of work, including writing a converter program, but it may be key to saving a project.

It is never too late in a project to improve your code.

Not all results are worth reporting. Nested for-loops have a tendency to produce zillions of results, most of which are very similar or not worth reporting. Reporting should be based on the research question, not on available results. Identify similar models/statistics/datasets before running your model and prune out unnecessary variations. Make your choice based on economic or econometric criteria. Use the computer to eliminate irrelevant results, by implementing your criteria in an analysis program. Create figures and LaTeX tables directly from MATLAB to save time and repetitive work.

13.4. Shortening a program

There are several reasons for which you want reduce the number of lines in a program. First and foremost, the number of errors in a program is usually proportional to its length. Shorter programs are also easier to read and understand, often they are also faster. The process of shortening a program makes you think about the structure of the code and usually results in better and more robust programs. Just like with an essay, it is good practice to read over a program after the first draft and to re-write pieces of code that are not precise, not obvious to understand and lengthy.

Some considerations for shortening a program

Eliminate redundancies. Programs often contain the remains of first ideas that are no longer needed.

Move duplicated code into loops or functions. We often duplicate code using copy and paste; this should be avoided, mostly because of the danger of inconsistencies when one copy is changed and the other one not. The best rule is to move every code that is executed more than once into a loop or a function, depending on the situation.

Eliminate debugging code. When testing a program, it is a good idea to add a few `disp()` and to check whether intermediate quantities are in a realistic range. Once a program has been sufficiently tested, eliminate these lines.

Simplify the logic. This includes simplifying logical conditions in the flow control and generally re-thinking complicated sections of your code. Most tasks, especially in data preprocessing and data matching, can be performed with just a few lines of code, given a good approach is chosen.

Rethink your for loops. The best approach for almost all for loops is to run them from 1 to N and to transform the loop variable according to the problem. Complicated ranges for loop variables are a common source of errors.

Add temporary variables. Especially when transforming a loop variable or when using complicated data structures, a temporary variable increases the readability. When you need to use `AllData{modelNumber}.testRestults.readingTest.firstTry` several times, introduce a temporary variable `myReadingTest`.

13.5. Exercises

Exercise 13.1. Download the files `debugtest.m` and `bad_fn.m` and debug them. Document all changes that you perform with a short comment.

Exercise 13.2 (*Optimizing Code)**.** Reconsider example 25: the problem was solved in different ways. For each of the following problems, write *three* solutions into one m-File. The first solution should be loop-centric, as in the "loop" example. The second should be vector-centric as in the "vectors" example. For the first two versions, you may only use operators and flow control, but no built-in function from MATLAB. As a third version, write an optimal solution using any suitable command, including complex built-in functions. Measure the execution time for each version.

(a) Calculate the sum of all squared numbers up to N.

(b) Calculate the sum of all odd numbers up to N.

(c) Produce a 10×10 multiplication table in matrix form.

(d) Calculate the mean of 100 uniformly distributed random numbers. Initialize the random number generator for each trial in the same way such that all three versions give exactly the same result.

(e) Calculate $\sum_{k=0}^{N} \frac{1}{2^k}$

14. Parallel computing

14.1. Introduction and concepts

14.1.1. Parallel hardware

Until a few years ago, parallel computing was left to a few experts in super computing centers and only used for modelling weather, climate or nuclear bombs. The situation today is different, and the main reason is technology. Until now, microprocessors roughly doubled in speed every 18-24 month[1]. Recently, this trend seems to have changed. With the advent of multi-core systems, processors still become increasingly powerful, but in number of cores rather than in terms of megahertz. To take full advantage of a modern processor, calculations need to be divided into *tasks* that can be processed independently and in parallel.

This trend has been reinforced by the advent of computing on graphics cards (CUDA computing). A modern graphics card contains up to 512 “pipelines”. These are not fully independent processors, but with some restrictions, each pipeline can perform one calculation per processor cycle, i.e. up to 512 parallel calculations.

14.1.2. Parallel computing, independence and communication

The main problem in parallel computing has nothing to do with computers; it would occur just the same if a pen-and-paper calculation were divided among 2 students. It is easy to divide a Monte Carlo experiment that involves trowing a coin 100 times: each student throws the coin 50 times. It is not so easy to parallelize the simulation of one path of an AR(1) process. Imagine one student calculating x_t. What should the second student do? In order to calculate x_{t+1}, she needs x_t. But this is only available once the first student has finished her calculation. At this moment the first student is idle to calculate x_{t+1}, thus the second student is permanently out of work.

The independence requirement is based on the simple observation that a calculation can only be performed once all its inputs are available. Splitting code into independent tasks is

[1]This observation is often quoted as Moore’s law, because Intel co-founder Gordon E. Moore published this observation (Moore 1965).

the main problem of parallel computing. There is no general rule to identify independent tasks. A way to automatically parallelize code does not (yet) exist.

14.1.3. Data parallel versus tasks parallel

Parallel programs can be distinguished into two classes:

Data parallel each task executes the same calculation on different data.
Example: calculation of t-statistics and p-values for each regression coefficient.

Task parallel each task executes a different calculation (on same or different data set).
Example: calculation of mean, mode and median.

14.1.4. Embarrassingly parallel problems.

Some data parallel problems are easy to parallelize, e.g. Monte Carlo simulations. Instead of running 1000 simulations on one machine, run 100 simulations each on ten machines. The same is true for all inherently independent problems such as scenario evaluations, grid search and many optimization algorithms.

14.2. Implementations of data parallel tasks in MATLAB

Parallel libraries. Many multivariate calculations are inherently parallel. As a simple example, take the sum of two $N \times N$ matrices. The calculation involves N^2 sums, which are all independent. Matrix calculations in MATLAB are handled by two standard libraries called BLAS (basic linear algebra subsystem) and LAPACK (linear algebra package). These libraries are already parallelized. To make full use of them, click on File/Preferences/General/Multithreading and select Enable . . . .

MATLAB's parfor. In many cases, it is sufficient to replace a `for`-loop with `parfor`.

Manual parallelization. Other problems have to be parallelized by clever thinking. As this topic is fairly new, there are no standard approaches for many problems. The challenge is to find independent pieces of code; these can be executed in parallel. A good place to start looking for parallelization opportunities are all loops.

Time series models. are hardest to parallelize. If there is autocorrelation, one cannot simply break a time series into to independent pieces – they are not independent. However, if the autocorrelation is low, one may break a time series of 1000 periods into two series of 550 periods each. The overlap can be used to match the parts.

14.3. Exercises

Exercise 14.1 (Parallel tasks in econometrics.)**.** Which of the following statistical quantities can be calculated independently? Produce a list that is grouped according to the following criteria: (1) all quantities that can be calculated immediately, (2) quantities that only require results from 1, (3) quantities that also require results from 2 and so on. Assume $X_{N\times 2}$ and $Y_{N\times 1}$ as given (subscripts indicate dimension).
List of quantities: mean of Y, median of Y, variance of Y, skewness of Y, kurtosis of Y, regression coefficient β_1, regression coefficient β_2, residuals, t-statistics for β_1, t-statistics for β_2.

15. MATLAB and databases

15.1. Introduction

MATLAB is not good at handling huge amounts of data. The whole dataset has to be loaded into the memory, even if one wants to work with a small subset. If a dataset is larger than 100MB, it is more efficient and convenient to set up a database.

There are two strategies for collaboration between MATLAB and a database. One can use the database to prepare smaller input files for MATLAB. This has the advantage that no changes are necessary to the MATLAB program at the price that one may soon be flooded with different data files.

The alternative is to link MATLAB to the database. This is a bit more difficult to set up, but it is much more efficient. One can query a database directly from MATLAB; the search result is stored in a variable and can be immediately used for calculations.

15.2. Setting up a database system

The most popular free database system is MySQL. It is available for many operating systems and can be downloaded from `www.mysql.com`. There are two useful tools that come with it: MySQL Administrator and MySQL Query Browser, for setting up and querying the database. In our example, we want to set up a database of option prices. MySQL, like most database programs, uses the SQL (standard query language) command standard.

Step 1: create a database
Before we can load the data, we need to create an empty database structure. SQL has a hierarchical structure. On the top is the *database* (normally, you just have one). Each database contains one or more *tables*. A table is a structure that contains the data, think of it as a questionnaire. Finally, the table is made up of *fields*, where each filed contains one piece of data. A field can be compared to one question in the questionnaire. Every filed is of one specific data type (mostly text or number, see below).

NOTE: double lines denote SQL-commands. Enter them in the MySQL query browser.

```
CREATE DATABASE optdb;
```

The first step is to create a new database. Note the semicolon at the end of each command.

```
USE optdb;                          Tell MySQL which database to use.
                                    Only needed once per session.
```

```
CREATE TABLE options (              A table is the structure for our data.
  date date NOT NULL,               The date of the price is of type date.
  expiry date NOT NULL,             Same for the expiry date.
  underlying char(10) NOT NULL,     A code for underlying (max 10 letters/numbers)
  type char(1) NOT NULL,            This will be P for put and C for call
  strike DECIMAL(14,6) NOT NULL,    The strike price.
  bid DECIMAL(14,6) NOT NULL,       Bid and ask prices: data type DECIMAL,
  ask DECIMAL(14,6) NOT NULL,       which is more precise than DOUBLE.
  INDEX date (date),
  INDEX expiry (expiry),            Define indices: these are tables
  INDEX underlying (underlying),    like the index of a book; they make
  INDEX type (type),                searching the database much faster.
  INDEX strike (strike),
  INDEX bid (bid),
  INDEX ask (ask),
) ENGINE=InnoDB DEFAULT CHARSET=latin1;
```

Step 2: load the data into the database

Most data comes in the form of a flat file, where each row contains one observation (or one *record*). Do not try to manipulate a data file before loading it into the database. Many spreadsheet programs, including MS Excel, cannot handle large data sets and may truncate the data.

```
LOAD DATA INFILE 'MyFile.csv'       Specify the full path to the file
  INTO TABLE options                Tell MySQL where to put the data
  FIELDS TERMINATED BY ','          A common format: comma separated values.
  (date, expiry, type,              Which fields to load. Correct here for a different
  underlying, strike, bid, ask);    order in the data file. Semicolon at end.
```

TIP: Step 1 and step 2 can also be performed via a graphical user interface in the MySQL Administrator. In any case is a good idea to save the SQL code to a text file in case you want to set up the database again (e.g. on a new computer).

Step 2a: document the database

Just like a program, you need to document your database. This includes the exact definition of each record, the data preparation and import procedure and possible symbols or codes that you use (e.g. codes for the underlying). To make comments in your SQL script, use the `#` sign (like `%` in MATLAB).

Step 3: play around in MySQL
Before making the link to MATLAB, verify that the import was successful and try a few SQL commands to make yourself comfortable with the syntax. (SQL commands are usually typed in ALL CAPS).

`SHOW DATABASES;`	Show all databases (probably 1).
`USE optdb;`	Use `opt` database.
`SHOW TABLES;`	Show all tables (probably 1).
`SHOW COLUMNS FROM options;`	Our database structure.

Step 4: the SELECT command
Searching and finding is performed using the `SELECT` command. This command consists of several elements. The asterisk (*) stands for "all fields in the table".

SELECT *what_to_display* **FROM** *table* **WHERE** *conditions* **ORDER BY** *fields*

`SELECT * FROM opt;`	All fields (*) from all records.
`SELECT * FROM opt LIMIT 30;`	Display only first 30 records.
`SELECT COUNT(*) FROM opt;`	A function: number of records.
`SELECT MIN(date) FROM opt;`	First date – `MAX()` finds last one.
`SELECT DISTINCT date FROM opt;`	List of all different dates.
`SELECT COUNT(DISTINCT date) FROM opt;`	How many different trade dates?
`SELECT date, strike, bid FROM opt;`	Display these three fields.
`SELECT * FROM opt WHERE date=’2003-08-14’;`	Search using `WHERE`.
`SELECT * FROM opt WHERE type=’P’ AND strike>100;`	Two conditions: `AND`.
`SELECT * FROM opt WHERE bid-ask < 1.0;`	Search for low bid-ask spread.
`SELECT * FROM opt WHERE datediff(expiry date)=30;`	Exactly 30 days to maturity.
`SELECT * FROM opt WHERE type=’C’ ORDER BY date;`	Sort results using `ORDER BY`.

Example: identify duplicates in a database
To get stated, we use `GROUP BY` to produce the data for a histogram of the strike prices:

```
SELECT strike, COUNT(*)  FROM opt GROUP BY strike
```

A duplicate is identified by the same date, expiry, strike and type, but possibly different prices. We group our data in all four categories and count the number of entries in each group. If it is more than one, we have identified a duplicate.

```
SELECT date, strike, expiry, type, COUNT(*)  FROM opt GROUP BY date,
    strike, expiry, type HAVING ( count(*)>1 )
```

15.3. Connecting MATLAB to the database

You need to install a special function called `mym` available from `sourceforge.net/projects/mym`. Put `mym` along with the accompanying libraries into a folder and add this folder to the MATLAB path using the command File/Set Path .... A new function called `mym()` becomes immediately available. The output argument of the function is the result of a MySQL query, where one field is stored in one variable.

NOTE: the commands in this subsection are again MATLAB commands.

`host='localhost';`	Or IP address of remote database.
`user='YourUsername';`	Define your settings.
`pwd='YourPassword';`	
`result=mym('open',host,user,pwd);`	Open the connection.
`result=mym('USE optdb');`	Use your database.
`qry='SELECT date,expiry,strike,bid,ask FROM opt'`	Define a query.
`[date,expiry,strike,bid, ask] = mym(qry);`	Execute query; result will be in the variables `date, strike` ...
`mym('CLOSE');`	Don't forget to close at the end.

A. A recap of matrix algebra

References: Gentle (very detailed),Greene Appendix A, Simon/Blume, The matrix cookbook, Wooldridge Appendix D, Verbeek Appendix A, Abadir/Magnus (very advanced), Härdle and Simar (2003) sections 2.2 and 9 (for spectral theory)

A.1. Matrices, vectors and scalars

Why matrix notation? Matrix notation has three advantages: it is concise, powerful and we have excellent computing tools for it. Matrices represent a two-dimensional array of numbers using just one symbol. This is a perfect representation for most economic data.

Definition 17. A *matrix* is a rectangular array of scalar numbers. A matrix with m rows and n columns is called $m \times n$-matrix. There are four different notations for matrices: a plain symbol, a symbol with subscript dimensions, one element with running indices and the full matrix:

$$A = A_{m,n} = (a_{ij}) = \begin{bmatrix} a_{11} & \cdots & a_{1j} & \cdots & a_{1n} \\ \vdots & \ddots & \vdots & & \vdots \\ a_{i1} & \cdots & a_{ij} & \cdots & a_{in} \\ \vdots & & \vdots & \ddots & \vdots \\ a_{m1} & \cdots & a_{mj} & \cdots & a_{mn} \end{bmatrix}$$

Note that there cannot be “holes” in a matrix. Thus one has to be very careful about missing entires in a data matrix. Missing data should never be replaced by a zero, as this would almost surely lead to a bias in the estimation.

A $m \times n$ matrix can be decomposed in m row vectors of dimension $1 \times n$ or n column vectors of dimension $m \times 1$:

Row vector $A(i,:) = (a_{i1}, \ldots, a_{ij}, \ldots, a_{in})$ **Column vector** $A(:,j) = \begin{bmatrix} a_{1j} \\ \vdots \\ a_{ij} \\ \vdots \\ a_{mj} \end{bmatrix}$

Conventions. The dimensions of a matrix A_{mn} are always quoted as **rows** $\times$ **columns**. The same is valid for the index of the element (a_{ij}): the first index is the row index, the second one the column index. This convention is also used by most software programs and programming languages, e.g. by MATLAB. In the case of *multivariate data*, one individual is usually represented in a row vector whereas one property is represented in a column vector. In the case of *panel data*, the first index is usually the time index: one column represents one individual over time, one row represents all individuals at one point in time.

$$\left.\begin{array}{l}\text{Dimension of } A_{mn}\\ \text{Index of } (a_{ij})\end{array}\right\}\ \textbf{rows} \times \textbf{columns}$$

A.2. Special matrices

- *Square matrix:* equal number of columns and rows, $m = n$.
- *Diagonal matrix:* all elements except along the main diagonal a_{ii} are zero.
- *Identity matrix* is a diagonal matrix with $a_{ii} = 1$. The identity matrix is the neutral element in matrix multiplication: any matrix multiplied with the identity matrix remains the same. There are several symbols for the $n \times n$ identity matrix: I, I_n and Id (for identity), the bold **1** and the German language E. MATLAB command: `eye(N)`. Example: $I_2 = \begin{bmatrix} \mathbf{1} & 0 \\ 0 & \mathbf{1} \end{bmatrix}$
- *Scalar matrix:* a diagonal matrix with the same value in all diagonal elements. a_{ii} are different from zero. Example: $\begin{bmatrix} 2 & 0 \\ 0 & 2 \end{bmatrix}$
- *Zero matrix:* a matrix with all elements equal to zero: $(a_{ij}) = 0$. Sometimes, the bold **0** is used as a symbol. The zero matrix is the neutral element of the matrix addition. MATLAB command: `zeros(N)`. Example: $\begin{bmatrix} 0 & 0 \\ 0 & 0 \end{bmatrix}$
- *Symmetric matrix:* a square matrix is symmetric if $a_{ij} = a_{ji}$. Properties of these matrices are very important, because the covariance matrix is a symmetric, square (and positive definite) matrix. A symmetric matrix is equal to its transposed: $A = A'$. Example for a symmetric matrix: $\begin{bmatrix} 1 & -2 & 3 \\ -2 & 7 & 5 \\ 3 & 5 & 0 \end{bmatrix}$
- Upper (lower) *triangular matrix:* a matrix with all elements below (above) the main diagonal zero. Example for an upper triangular matrix: $\begin{bmatrix} 1 & 2 & 3 \\ \mathbf{0} & -7 & 5 \\ \mathbf{0} & \mathbf{0} & 1 \end{bmatrix}$
- *Projection matrix:* a square matrix for which $B \cdot B = B$, also called *idempotent matrix.* Example: the identity matrix or $\begin{bmatrix} -2 & 3 \\ -2 & 3 \end{bmatrix}$
- *Orthogonal matrix:* a square matrix, that fulfills $A' = A^{-1}$. This implies that $AA' = Id$ and that the row/column vectors are pairwise orthonormal.

A.3. Matrix calculation rules

Many of the following rules are also valid for vectors, as vectors can be seen as $n \times 1$ viz. $1 \times m$ matrices.

Addition and subtraction. Matrices are added (subtracted) element by element:

$$A + B = C = (c_{ij}) = (a_{ij} + b_{ij})$$

Only matrices of the same dimension can be added (subtracted). This implies that it is not possible to add a scalar to a matrix.

$$\underline{\text{Properties:}} \quad \begin{aligned} &A + B = B + A \\ &A + (B + C) = (A + B) + C \\ &A + \mathbf{0} = A \end{aligned}$$

> `MATLAB note:` In MATLAB, it is possible to add a scalar x to a matrix A with the command `A+x`. In this case, the scalar x is added to *every* element of the matrix A.

Scalar multiplication A matrix is multiplied by a scalar by multiplying each element of the matrix with this scalar:

$$\alpha A = \begin{bmatrix} \alpha a_{11} & \alpha a_{12} & \dots & \alpha a_{1n} \\ \alpha a_{21} & \alpha a_{22} & \dots & \alpha a_{2n} \\ \vdots & \vdots & \ddots & \vdots \\ \alpha a_{n1} & \alpha a_{n2} & \dots & \alpha a_{nm} \end{bmatrix}$$

$$\underline{\text{Properties:}} \quad \begin{aligned} &\alpha(A + B) = \alpha A + \alpha B \\ &\alpha A = A\alpha \end{aligned}$$

Scalar product. The scalar product (or "dot" product) is only defined for one row vector times one column vector of same number of elements.

$$a'x = a_1x_1 + a_2x_2 + \dots + a_nx_n$$

An alternative notation with two column vectors is: $\langle ax \rangle = a'x$

$$\underline{\text{Properties:}} \quad \begin{aligned} &a'x = x'a \\ &a'x = 0 \text{ for orthogonal vectors} \end{aligned}$$

> `MATLAB note:` It can be quite confusing which vector to transpose to obtain the vector product. Alternatively, use `dot(x,y)`.

Matrix multiplication. The product of the matrices $A_{m,n}$ and $B_{n,p}$ is a matrix $C_{m,p}$ with

$$c_{ij} = \sum_{k=1}^{n} a_{ik} b_{kj} \qquad i = 1, \ldots, m \quad j = 1, \ldots, p$$

c_{ij} can be seen as the scalar product of the i^{th} row vector and the j^{th} column vector: $c_{i,j} = A(i,:)B(:,j)$.

Matrices have to be of matching dimensions, i.e. the number of columns in A must be equal to the number of rows in B. Note that even if both exist, generally $AB \neq BA$. This is especially true for vectors, where $x' \cdot x$ is a scalar, whereas $x \cdot x'$ is a matrix. A special case of matrix multiplication is multiplying a column vector with a row vector, which gives a matrix.

$$\begin{aligned} \underline{\text{Properties:}} \quad & A \cdot Id = Id \cdot A = A \\ & A(B+C) = AB + AC \\ & (B+C)A = BA + CA \\ & A(BC) = (AB)C \\ & AB \neq BA \text{ (generally)} \end{aligned}$$

The vec operator. This operator *vec*torizes a matrix into a vector by stacking the column-vectors. Thus one can apply vector-only operations to matrices. In the 2×2 case, the *vec* operator looks like this:

$$vec \begin{bmatrix} 1 & 2 \\ 3 & 4 \end{bmatrix} = \begin{pmatrix} 1 \\ 3 \\ 2 \\ 4 \end{pmatrix}$$

`MATLAB note:` The MATLAB command for $vec(A_{m,n})$ is `reshape(A,n*m,1)` or `A(:)`

$$\begin{aligned} \underline{\text{Properties:}} \quad & vec(A+B) = vec(A) + vec(B) \\ & vec(\alpha A) = \alpha vec(A) \\ * \quad & Tr(A'B) = vec(A)' \cdot vec(B) \\ * \quad & vec(ABC) = [C' \otimes A] vec(B) \end{aligned}$$

Equality. Equality is defined element-by-element. This implies that equality is only defined for matrices of the same dimension.

$$A_{m,n} = B_{m,n} \Leftrightarrow a_{ij} = b_{ij} \quad \forall i,j$$

Rank of a matrix. In any matrix, the number of linear independent row vectors equals the number of linear independent column vectors. If the matrix is not square, this implies that there must be some linear dependent vectors in the larger dimension.

Definition 18. The rank of a matrix is the number of linear independent column/row vectors. As discussed above $rk(A_{m,n}) \leq \min(m,n)$. Furthermore, a matrix is called *regular* (or is said to have *full rank*), if $rk(A_{m,n}) = \min(m,n)$.

The rank crucial for the solvability of a system of equations, see "inverse" below. However, the rank criterion has one disadvantage: There are many real-world cases with random errors, in which matrices are "close to singular". In such a case, it is better to consider the condition number, see page 179.

$$\begin{aligned} &\underline{\text{Properties:}} \quad rk(A+B) \leq rk(A) + rk(B) \\ &rk(AB) = \min(rk(A), rk(B)) \\ &rk(A'A) = rk(A) \qquad \text{(Useful for OLS regularity)} \end{aligned}$$

Example 26. Dummy variable trap. If there is a constant term (=1), we cannot have a dummy variable for "male" and for "female", because $male + female = 1$ which is linear dependent from the constant.

Inverse of a matrix. The inverse of a square matrix is defined in analogy to the inverse of a scalar. For any scalar number a, the inverse a^{-1} fulfills the equation $a \cdot a^{-1} = 1$. For a matrix A, this is generalized to

$$AA^{-1} = Id.$$

Theorem 2. *The inverse of a matrix exists if and only if the matrix is regular. Proof: Simon-Blume, p. 166.*

Calculation of the inverse: see determinants. In practice, one tries to avoid calculating the inverse of a matrix, because it is computationally intensive and max introduce imprecisions, see the example below.

$$\begin{aligned} &\underline{\text{Properties:}} \quad (A^{-1})^{-1} = A \\ &(AB)^{-1} = B^{-1}A^{-1} \\ &AA^{-1} = A^{-1}A = Id \\ &(\alpha A)^{-1} = \tfrac{1}{\alpha}A^{-1} \\ &(A^{-1})' = (A')^{-1} \end{aligned}$$

Example 27. We use the inverse to solve a system of equations. As the inverse exists only for square matrices with full rank, this approach works only for exactly identified systems.

$$\begin{aligned} y &= Ax \\ A^{-1}y &= x \end{aligned}$$

`MATLAB note:` Gaussian elimination is a fast and precise alternative to the inverse. MATLAB provides the special backslash operator for this. In our example: `x=y\A`

Transposed of a matrix The transposed of a matrix obtains by exchanging its rows and columns:

$$A_{m,n} = (a_{ij}) \Rightarrow A'_{n,m} = (a_{ji}) \quad .$$

Note that the number of rows and columns are exchanged between a matrix and its transposed. Transposing converts a row vector into a column vector and vice versa. This is often used to shorten the notation of a column vector: $x = (x_1, x_2, x_3)'$.

Properties:
$(A')' = A$
$Id' = Id$
$(A + B)' = A' + B'$
$(\alpha A)' = \alpha A'$
$(AB)' = B'A'$
A symmetric $\Rightarrow A = A'$
$rk(A'A) = rk(A)$ (A nonsquare, full rank $\rightarrow$, $A'A$ *or* AA' full rank)

Trace of a matrix The trace of a square matrix is the sum of the diagonal elements. It has a lot of convenient properties.

$$Tr(A_{n,n}) = \sum_{i=1}^{n} (a_{ii})$$

Properties:
$Tr(Id_n) = \sum_{i=1}^{n} 1 = n$
$Tr(A) = Tr(A')$
$Tr(A + B) = Tr(A) + Tr(B)$
$Tr(AB) = Tr(BA)$ (this gives rise to:)
$Tr(ABC) = Tr(CAB) = \ldots$ (any rotation gives the same trace)
$Tr(\alpha A) = \alpha Tr(A)$
$Tr(\alpha) = \alpha$ (This simple identity gives rise to:)

Determinant of a matrix. The determinant is a scalar function of a square matrix A. It is nonzero if and only if the matrix is regular (Proof: Simon-Blume, p. 193). One can say that the determinant *determines* whether a square matrix is singular or not. Later we will see that the determinant is also the product of the characteristic roots of the matrix.

There are different ways how to compute the determinant. One way is the recursive definition. It starts with $n = 1$. A scalar is its own determinant: $\det \alpha = \alpha$. For larger matrices, the following rules apply: any row i can be chosen. For every element a_{ij} in this row, we can define a cofactor C_{ij} as

$$C_{ij} = (-1)^{(i+j)} M_{ij}$$

where M_{ij} is the so-called minor, the determinant of a matrix A_{ij}, which obtains if the i-th row and the j-th column of A are deleted. The determinant is then

$$\det A = \sum_{i=1}^{n} a_{ij} C_{ij}.$$

Note that if A is of dimension n, then A_{ij} is of dimension $(n-1)$. To calculate $M_{ij} = \det A_{ij}$, one can re-apply this rule and obtain a matrix of dimension $(n-2)$ and so on until one can use $\det \alpha = \alpha$.

The determinant of a 2×2 matrix is the product of the diagonal elements minus the product of the off-diagonal elements:

$$\begin{vmatrix} a_{11} & a_{12} \\ a_{21} & a_{22} \end{vmatrix} = a_{11}a_{22} - a_{21}a_{12}$$

`MATLAB note:` MATLAB uses triangular matrices to calculate the determinant of A. First, A is decomposed into a lower and an upper triangular matrix $A = LU$. Next, MATLAB uses the identity $\det(AB) = \det(A)\det(B)$ and the fact that the determinant of a triangular matrix is exactly the product of its diagonal elements: $\det L = \prod_{i=1}^{n} l_{ii}$.

Properties:	$\det \alpha = \alpha$	
	$\det A' = \det A$	
	$\det(A \cdot B) = (\det A)(\det B)$	
	$\det(A + B) \neq \det A + \det B$	in general
	$\det(A^{-1}) = \frac{1}{\det A}$	
	$\det A = \prod_{i=1}^{n} a_{ii}$	for A triangular or diagonal

Quadratic forms. A matrix $A_{n \times n}$ can be multiplied with a vector $\mathbf{x}_{n \times 1}$ and its transposed to yield a scalar. This expression is called a quadratic form:

$$x'Ax$$

Positive definiteness. A matrix is called positive (semi-) definite if the quadratic form of every vector $\mathbf{x}$ is positive (positive or zero):

$$x'Ax > (\geq)0$$

Analogy to the scalar case: Take two real numbers a and x and construct the equivalent of a quadratic form:

$$xax = ax^2$$

This "quadratic form" is positive iff $a > 0$ and negative iff $a < 0$, regardless of the value of x (i.e. for all values of x). In the matrix world, there is an additional possibility: a matrix can be indefinite.

Properties of pos.	$x_{ii} > 0$ (diagonal elements strictly positive)
definite matrices	$\det A^{-1}$ exists and is also pos. def.
	$X'X$ and XX' are at least positive semidefinite
	for $X_{n\times k}$: $X'X$ is positive definite

A.4. Frequently used tricks with vectors

- Sum of elements: $\mathbf{x}'\mathbf{1} = \sum x_i$,
  where $\mathbf{1}$ is a vector of ones.
- Sum of squares: $\mathbf{x}'\mathbf{x} = \sum x_i^2$
- Variance-Covariance matrix: $= X'X$
  Where X is the de-meaned data matrix (individuals in rows, variables in columns).

A.5. Spectral theory

A.5.1. Eigenvalues and eigenvectors

Spectral theory is the theory of eigenvalues and eigenvectors[1]. Spectral theory is the main tool to understand the structure of a matrix – and more general of linear operators. The main idea behind it is to split the matrix into simple pieces and to analyze each piece separately.

[1]The famous German mathematician David Hilbert coined the terms "Eigenwert" and "Eigenvector" in 1904. The English translation "proper value" was used for some time, but did not prevail.

Definition 19. A scalar value λ is called an *eigenvalue* of the $n \times n$ matrix A if there exists a nonzero vector $\mathbf{x}$ such that

$$A\mathbf{x} = \lambda\mathbf{x} \tag{A.1}$$

Definition 20. A vector $\mathbf{x}$ that satisfies (A.1) is called *eigenvector.*

To find the eigenvalues, one has to solve (A.1) for *any* possible $\mathbf{x}$:

$$\begin{aligned} A\mathbf{x} &= \lambda\mathbf{x} \\ (A - \lambda Id)\mathbf{x} &= 0 \end{aligned} \tag{A.2}$$

Equation (A.2) is a system of equations which has a nontrivial solution only if the coefficient matrix $(A - \lambda Id)$ is singular. This is equivalent to the statement $\det(A - \lambda Id) = 0$. In the 2×2 case, $\det(A - \lambda Id)$ takes the following form:

$$\begin{aligned} \det \begin{bmatrix} a_{11} - \lambda & a_{12} \\ a_{21} & a_{22} - \lambda \end{bmatrix} &= (a_{11} - \lambda)(a_{22} - \lambda) - a_{12}a_{21} \\ &= \lambda^2 - \lambda(a_{11} + a_{22}) + a_{11}a_{22} - a_{12}a_{21} \end{aligned} \tag{A.3}$$

The polynomial in A.3 is called *characteristic polynomial.* Finding the eigenvalues λ_i is equivalent to finding all roots of the characteristic polynomial. It is easy to see that for a $n \times n$ matrix, the characteristic polynomial is of degree n. This implies that such a matrix has n eigenvalues (although some could be identical or complex). If a value λ_i occurs more than once – k times – as an eigenvalue, it is said to have a *multiplicity* of k.

Definition 21. The set of all eigenvalues of a matrix A is called its *spectrum.* It is sometimes denoted as $\sigma(A)$.

Example 28. The characteristic polynomial of Id_2 is $(1-\lambda)(1-\lambda)$. It has two eigenvalues, both are 1. Its spectrum is $\sigma(Id_2) = \{1, 1\}$.

To find the eigenvector that corresponds to a certain eigenvalue, one has to solve the system of equations $(A - \lambda_i Id)\mathbf{x}_i = 0$. The eigenvector $\mathbf{x}_i$ may have complex elements. Note that if $\mathbf{x}_i$ is an eigenvector, then $\alpha\mathbf{x}_i$ is an eigenvector, as well. The convention is to normalize all eigenvectors to a length of 1.

Eigenvectors for distinct eigenvalues are always linearly independent (Proof: Simon-Blume, sect. 23.9). In the case of multiplicity, there still may exist linearly independent eigenvectors (e.g. for the identity matrix in example 28). See exercise 7.1.

Properties: $TrA = \lambda_1 + \lambda_2 + \cdots + \lambda_n = \sum_{i=1}^n 1\lambda_i$
$\det A = \lambda_1 \cdot \lambda_2 \cdot \cdots \cdot \lambda_n = \prod_{i=1}^n \lambda_i$
A triangular: $\lambda_i = a_{ii}$ (also valid for diagonal matrices)
$\lambda_i \neq \lambda_j \rightarrow \mathbf{x}_i\mathbf{x}_j = 0$ (Eigenvectors are linearly independent)
A singular $\rightarrow$ at least one eigenvalue $= 0$

`MATLAB note:` The command `l=eig(A)` produces a vector `l` with all eigenvalues and `[X,D]=eig(A)` produces a matrix `X` whose columns are the eigenvectors and a diagonal matrix `D` whose elements are the eigenvalues.

A.6. Applications of spectral theory

A.6.1. Matrix decompositions

Diagonalization

It would be very practical to decompose a $n \times n$ matrix A in the following way:

$$A = PDP^{-1} \tag{A.4}$$

where D is a diagonal matrix and P is any general matrix. A^2 could then be calculated as $A^2 = (PDP^{-1})^2 = PDP^{-1}PDP^{-1} = PD^2P^{-1}$ and generally we could write

$$A^n = PD^nP^{-1} \tag{A.5}$$

This is practical, because it is very easy to compute D^n, e.g. in the 3×3 case:

$$D^n = \begin{bmatrix} (d_{11})^n & 0 & 0 \\ 0 & (d_{22})^n & 0 \\ 0 & 0 & (d_{33})^n \end{bmatrix} \tag{A.6}$$

Proposition 1. Any square matrix A with distinct eigenvalues can be diagonalized in the form of (A.4) with P the matrix of eigenvectors $P = [\mathbf{x}_1, \mathbf{x}_2, \ldots, \mathbf{x}_n]$ and D a diagonal matrix of eigenvalues.

Proof.

$$\begin{aligned} AP &= A[\mathbf{x}_1 \cdots \mathbf{x}_n] \\ &= [A\mathbf{x}_1, \cdots A\mathbf{x}_n] \\ &= [\lambda_1\mathbf{x}_1 \cdots \lambda_n\mathbf{x}_n] \\ &= [\mathbf{x}_1 \cdots \mathbf{x}_n] \begin{bmatrix} \lambda_1 & \cdots & 0 \\ \vdots & \ddots & \vdots \\ 0 & \cdots & \lambda_k \end{bmatrix} \\ &= PD \end{aligned} \tag{A.7}$$

We denoted the matrix of the eigenvalues as D, because it is – as desired – a diagonal matrix. Next we multiply (A.7) with P^{-1} from the right to get expression (A.4).

It remains to be shown that P^{-1} exists. This is the case, if P is regular, which means that all eigenvectors are linearly independent. This is exactly the case if all eigenvalues are distinct. See also exercise 7.1.

Symmetric matrices. Symmetric matrices have only real eigenvalues. Furthermore, even if some eigenvalues may not be distinct, P is always invertible. (Proof: Simon-Blume, sect 23.7). Thus, all symmetric matrices can be diagonalized. One can show, that P is orthogonal for symmetric matrices:

$$\begin{aligned} A &= A' \\ PDP^{-1} &= \left(PDP^{-1}\right)' \\ PDP^{-1} &= \left(P^{-1}\right)' D'P' \end{aligned} \tag{A.8}$$

Notice that $D = D'$. Comparing the coefficients, we find that P is orthogonal: $P' = P^{-1}$. Thus we can write (A.4) also as

$$A = PDP^{-1} = PDP' \tag{A.9}$$

Notation. Some books write $D = PAP^{-1}$ instead of $A = PDP^{-1}$, which is equivalent. Matrix diagonalization is sometimes referred to as PAP-decomposition.

Square root of a matrix

For scalars, the square root is well defined: $\sqrt{a}\sqrt{a} = a$ and there are normally two solutions (one positive, one negative). We can use (A.5) to calculate $A^{1/2}$. Note that this approach will give complex solutions if there are negative eigenvalues.

Functions of a matrix

It is possible to define standard functions like exp() or log() with a matrix argument. These functions can then be evaluated via their Taylor series, using the PDP decomposition. An alternative (faster) route is to apply the function directly to the PDP decomposition.

Example: functions of a matrix

`A=rand(3)`	Produce some random matrix.
`log(A)`	This is the element-by-element logarithm.
`B=logm(A)`	This is the matrix logarithm.
`expm(B)`	Applying the matrix exponential produces the original matrix.
`[X,D]=eig(A)`	Perform a PDP decomposition.
`X*D*inv(X)`	Produce original matrix.
`X*log(D)*inv(X)`	For the diagonal matrix D, we only need `log()`, not `logm()`.

Choleski decomposition

In many cases, we are not so much interested in $A^{1/2}$, but in a decomposition $A = RR'$, with R being an (upper) triangular matrix. For symmetric, positive-definite matrices (e.g. all covariance matrices), R always exists. The calculation is rather tedious, but MATLAB can do it for us with `chol(A)`.

Application. The Choleski decomposition can be used to produce correlated random numbers (most computer systems can only produce uncorrelated ones). Suppose $\Sigma = RR'$. Produce a random vector $\mathbf{y} = R\mathbf{z}$ with $\mathbf{z} \sim N(0, Id)$. Then $\mathbf{y} \sim N(0, \Sigma)$.

Proof: $V(\mathbf{y}) = E\left[(\mathbf{y} - \mu_y)(\mathbf{y} - \mu_y)'\right] = E\left[\mathbf{y}\mathbf{y}'\right] = E\left[R\mathbf{z}\mathbf{z}'R'\right] = RIdR' = RR' = \Sigma$

Singular value decomposition (SVD)

The Choleski decomposition is only available for symmetric, square and positive-definite matrices. Although all covariance matrices fall in this category, it would be desirable to have a matrix decomposition that allows more general input. This is the singular value decomposition (SVD).

Proposition 2. One can decompose any matrix $A_{n,m}$ with rank $p \leq min(m, n)$ as

$$A_{n\times m} = U_{n\times p}\Sigma_{p\times p}V'_{p\times m} \tag{A.10}$$

U, V orthogonal matrices containing the first p eigenvectors of AA' and $A'A$ respectively and Σ a diagonal matrix containing the square roots of the first p eigenvalues of AA'.

Proof (sketch). Matrix diagonalization is only possible for square matrices. However, the matrices AA' and $A'A$ are square and furthermore symmetric such that the decomposition (A.9) is possible. We verify that this is the case:

$$\begin{aligned} AA' &= U\Sigma V'V\Sigma'U' = U\Sigma^2U' \\ A'A &= V\Sigma U'U\Sigma'V' = V\Sigma^2V' \end{aligned} \tag{A.11}$$

By comparing this result to (A.9), we find that U and V are indeed the eigenvectors of AA' and $A'A$. Furthermore, $\Sigma^2 = D$ or $\Sigma = \sqrt{D}$ from (A.9). This is indeed the diagonal matrix of the square roots of the eigenvalues. To calculate $\sqrt{D}$, see (A.6).

A.6.2. Condition number

In chapter 2, we have stated that the determinant of a singular matrix is zero. With real data, we will hardly ever run across perfect linear dependence, so one could argue that a determinant "close to" zero is a good criterion for singularity. Now, consider the matrix $10^{-3} \cdot Id_4$. Its determinant is 10^{-12}, which is definitely close to zero, however this matrix is perfectly invertible.

To find a better criterion, we notice a singular matrix has at least one eigenvalue which is zero. Furthermore, matrices that are close to being singular, have at least one eigenvalue which is very small compared to the other eigenvalues. This can be used to define a measure for linear dependence that is more precise than the determinant.

Definition 22. The condition number of a matrix A is defined as

$$k(A) = \frac{\sqrt{\lambda_{max}}}{\sqrt{\lambda_{min}}}$$

Where $\lambda_{max,\,min}$ are the largest/smallest eigenvalue of A. A rule of thumb states that a condition number of more than 15 indicates a possible problem and a condition number of more than 30 indicates a serious problem with colinearity. Note that our matrix $10^{-3} \cdot Id_4$ has a condition number of 1 – the smallest possible one.

B. A recap of complex algebra

The starting point of complex algebra is the definition of a new symbol for the square root of -1:

$$i = \sqrt{-1} \tag{B.1}$$

The powers of i are:

$$i^2 = -1 \qquad i^3 = -i \qquad i^4 = 1 \tag{B.2}$$

Complex numbers consist of a real part a and an imaginary part b

$$z = a + ib \tag{B.3}$$

If only one of the two is needed, use `a=real(z)` and `b=imag(z)`.

The two parts are summed individually

$$z_1 + z_2 = (a_1 + ib_1) + (a_2 + ib_2) = (a_1 + a_2) + i(b_1 + b_2) \tag{B.4}$$

Complex numbers can alternatively be represented in polar coordinates

$$z = (|z|, \phi) \tag{B.5}$$

with $z^2 = a^2 + b^2$ and

$$\varphi = \arg(z) = \begin{cases} \arctan(\frac{b}{a}) & \text{if } a > 0 \\ \arctan(\frac{b}{a}) + \pi & \text{if } a < 0 \text{ and } b \geq 0 \\ \arctan(\frac{b}{a}) - \pi & \text{if } a < 0 \text{ and } b < 0 \\ \frac{\pi}{2} & \text{if } a = 0 \text{ and } b > 0 \\ -\frac{\pi}{2} & \text{if } a = 0 \text{ and } b < 0 \end{cases} \tag{B.6}$$

We usually call $|z|$ the absolute value and ϕ the argument (sometimes phase) of the complex number. The argument is sometimes denoted as $\phi = \arg(z)$. In MATLAB, we can easily calculate absolute value and argument by using `abs()` and `angle()`.

The Cartesian representation is obtained by:

$$a + ib = |z| \cos \phi + i|z| \sin \phi \tag{B.7}$$

Notice that the argument ϕ is multivalued. For example, the above equation would produce the same result of $\zeta = \phi + 2\pi$. As the argument is not unique, it is customary to restrict it to the interval $[-\pi, \pi]$.

An advantage of the polar representation is the easy calculation of powers and therefore polynomials.

$$z^n = (|z|^n, n\phi) \tag{B.8}$$

The above equation can also be used to calculate the square root of a complex number.

Similarly, we can calculate the logarithm of a complex number

$$\log(z) = \log(|z|) + i\phi \tag{B.9}$$

A famous identity discovered by Leonhard Euler is:

$$e^{i\phi} = \cos \phi + i \sin \phi \tag{B.10}$$

This leads, however, to some ambiguity, which plagues the characteristic function of Heston-like models (for more details, see Lord and Kahl (2008)):

$$e^{i(\phi+2\pi)} = \cos(\phi + 2\pi) + i \sin(\phi + 2\pi) = e^{i\phi} \tag{B.11}$$

B.1. Exercises

Exercise B.1. Using the symbolic mathematics toolbox, calculate the first six elements of the Taylor series of (a) $e^{i\phi}$ and (b) $\cos \phi + i \sin \phi$. Compare the results.

Exercise B.2. Verify numerically that $e^{i\phi} = \cos \phi + i \sin \phi$ by calculating the difference of the two terms for 100 values on the interval $[-2\pi, 2\pi]$

Exercise B.3. (a) Produce a function called `cplot` for plotting a complex vector. The plot should consist of two fields, the upper showing the real part and the lower showing the imaginary part. *Hint:* use the command `subplot`. (b) Produce a cplot for $\ln z$ with $Re(z) = Im(z) = 1 : 100$. (c) Produce a cplot for z with $Re(z) = 1 : 100$ and $Im(z) = cos(1 : 100)$. (d) Using z from (c), run the command `plot(z)`. Interpret the results.

C. An OLS refresher

The OLS estimator as a good illustration of matrix algebra applied to econometrics. The ordinary least squares (OLS) problem is the simplest regression problem in econometrics. We give here only a short overview. For details see my script "Linear Models and Variance analysis" or (Greene 2002) or (Verbeek 2004). The model is based on four assumptions. The notation is n=number of observations, k=number of variables.

MLR.1 Linearity in parameters
The regression model takes the form of $y = \beta_0 + \beta_1 x_1 + \beta_2 x_2 + \cdots + \beta_k x_k + \varepsilon$.
In matrix notation, this becomes (note that X also contains a column of 1s to catch β_0)

$$\mathbf{y}_{n\times 1} = X_{n\times(k+1)}\beta_{(k+1)\times 1} + \varepsilon_{n\times 1} \tag{C.1}$$

MLR.2 Zero conditional mean of errors: $E(\varepsilon|X) = 0$

MLR.3 Random sampling: this is implied in the fact that we have $\mathbf{y}_{n\times 1}$ and $X_{n\times(k+1)}$

MLR.4 No perfect colinearity: $rk(X) = k + 1$

Under the assumptions **ML.1** to **MLR.4**, the OLS estimator is an unbiased estimator for β:

$$\widehat{\beta} = (X.'X)^{-1}X.'\mathbf{y} \tag{C.2}$$

A common measure for the explanatory power of a regression is the coefficient of determination (or R^2). It is the sum of squared residuals (SSR) divided by the total sum of squares (SST).

$$R^2 = 1 - \frac{SSR}{SST} = 1 - \frac{\sum(y_i - \hat{y}_i)^2}{\sum(y_i - \bar{y})^2} \tag{C.3}$$

To allow for testing, we need to introduce two additional assumptions:

MLR.5 Homoskedasticity and no serial correlation: $Var(\varepsilon|X) = \sigma^2 Id$

MLR.6 Normality of errors. Conditional on X, ε_i are i.i.d. (independent and identically distributed) as: $\varepsilon_{n\times 1} \sim N(0, \sigma^2 Id)$

MLR.5 ensures that OLS is the BLUE (best linear unbiased estimator) and makes it possible to estimate the variance of ε. An estimator for σ^2 is:

$$\hat{\sigma}^2 = \frac{\hat{\varepsilon}'\hat{\varepsilon}}{n-k-1} \tag{C.4}$$

Note that the result of the regression, $\hat{\beta}$ is a random variable. Under the assumptions **MLR.1** through **MLR.6**, its conditional distribution is $\hat{\beta}|X \sim N(\beta, \sigma^2(X'X)^{-1})$. Thus we can calculate the conditional standard error of $\hat{\beta}_j$,

$$se(\hat{\beta}_j|X) = \sqrt{var(\hat{\beta}_j|X)} = \hat{\sigma}\sqrt{((X'X)^{-1})_{jj}} \tag{C.5}$$

and use it to devise a test statistic for the individual components of β.

$$\frac{\hat{\beta}_j - \beta_j}{se(\hat{\beta}_j)} \sim t_{n-k-1} \tag{C.6}$$

The standard test is the test for the null hypothesis whether $\hat{\beta}_j$ is significantly different from zero. For this test, we set $\beta_j = 0$.

The OLS estimator is the solution of a least squares problem: β minimizes the sum of squared error terms ($\varepsilon'\varepsilon = \varepsilon_1^2 + \cdots + \varepsilon_n^2$) in (C.1). Using $\varepsilon = \mathbf{y} - X\beta$ we can give an alternative formulation of the OLS problem. We will recur to this definition in chapter 10.

$$\hat{\beta} = \arg\min_{\beta} \varepsilon'\varepsilon \; = \; \arg\min_{\beta} \, (\mathbf{y} - X\beta)' \, (\mathbf{y} - X\beta) \tag{C.7}$$

D. Practical issues

D.1. MATLAB symbols

Squared brackets []

- Create vectors, matrices, e.g. `v=[4 5 6]`
- Empty vector, e.g. `x=[]`

Round brackets ()

- In equations, e.g. `(x+1)^2`
- Argument of a function, `log(x)`
- Indices of vectors, e.g. `l=v(1)`

Curly braces {}

- Cell arrays.

Semicolon ;

- Next row e.g. `M=[1 2; 3 4]`
- Suppress output, e.g. `x=y+z;`

Colon :

- Create a vector $x_{min} : x_{max}$ Example: `1:4` = `[1 2 3 4]`
- With step size $x_{min} : \Delta : x_{max}$ Example: `1:2:5` = `[1 3 5]`
- Access whole row/column of a matrix. Example: `A(1,:)` for row 1, `A(:,3)` for column 3.

Comma ,

- Next command, e.g. `x=1, y=2`
- Separate arguments of a function, e.g. `regress(Y,X)`
- Next element in a row vector, e.g.`x=[1, 2, 3]`. Can be omitted.

Dot .

- Decimal point `x=1.5`
- Dotted (i.e. element-by-element) operator, e.g. `vec1.*vec2`
- Fields in structures, e.g. `regr.r2`

Three dots ...

- Command continues in next line

Plain apostrophe '

- Complex conjugate e.g. `A'`
- Quotation marks for text strings, e.g. `'This is a text'`
  TIP: Quotations inside quotations use 2 quotation marks, e.g. `'I did the homework ''myself''.'`
- Quotation marks for filenames, e.g. `load('filename.mat')`
- Quotation marks for symbolic equations, e.g. `g=sym('1/phi+1=phi')`

At-sign @

- Function handles and anonymous functions

And-sign &, vertical line |

- Logical "and", logical "or"

D.2. MATLAB graphics

Use the plot symbol (▾) in the workspace to create a plot from data. To change a plot, enter `plottools` in the command window or click on the □ symbol in the plot window.

A few very useful commands for plots

`subplot(n,m,i)`	Produce n rows $\times$ m columns of plots in one figure. This plot will be number i, counted from top left to bottom right.
`hold on`	Overlay two plots. To start a from scratch, use `hold off`
`axis(xmin,xmax,ymin,ymax)`	Rescale a plot.

A simple plot

`a=randn(100,1); b=randn(100,1);`	Create two data series.
`a=cumsum(a); b=cumsum(b);`	A simple brownian motion.
`plot(a)`	For one plot, we don't need x-values.
`plot(a,b)`	Plot a against b (did we want this?)
`n=length(a)`	Obtain number of data points.
`plot(1:n,a,'k-',1:n,b,'r-.')`	Two curves in one plot. `'k-'` stands for "black, solid line" and `'r-.'` for "red, dash-dotted". See `help plot` for more settings.
`legend('Stock01','Stock02')`	Add legend in same order as data.

Time series

Imagine the above data are monthly observations for the years 2000 to 2007 (i.e. from the beginning of 2000 to the beginning of 2008).To get a plot with nice labels on the x-axis, use `datetick`. Assume `a, b, n` are still in the workspace.

`t=linspace(2000,2008,100)`	Get the fractional year for each of the 100 observations.
`x=datenum(t,1,1);`	Convert the fractional year to date number.
`plot(x,a,'b-',x,b,'r-.')`	Plot with date numbers; x-axis looks strange.
`datetick('x','mmmyy')`	Get correct x-ticks.
`datetick('x','QQ-YY')`	Alternative formatting.

Compare values

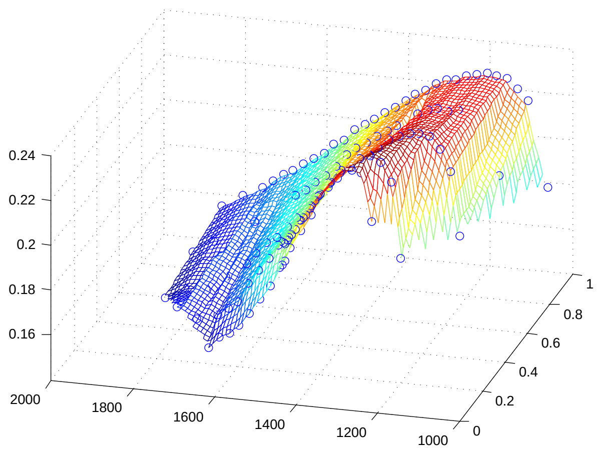
`x=1:10;`	Create x-values
`y_hat=2*x-2;`	Create estimated y-values
`y=2*x-2+randn(1,10);`	Create observed y-values
`plot(x,y,'+',x,y_hat,'o')`	Compare both: estimation is a circle (`'o'`), observation is a cross (`'+'`).
`legend('Observed','Estimated')`	Add legend in same order as data.

3D plotting plus interpolation: the implied volatility surface

The problem when plotting the implied volatility surface is the fact that the data is not regular: for short maturities, there are usually more strikes available than for long maturities. To overcome this problem, we interpolate the data.

`load iv.txt`	Load SP500 IV data for Jan 2, 2000
`strike=iv(:,2);`	Data format: strike, ...
`mat=iv(:,1);`	maturity, ...
`impvol=iv(:,3);`	implied volatility
`plot(mat,strike,'o')`	The data is not evenly spaced.
`Mmesh = min(mat):0.02:max(mat);`	Make an x-mesh ...
`Smesh = min(strike):10:max(strike);`	and a y-mesh for interpolation.
`[XI, YI] = meshgrid(Mmesh, Smesh);`	Produce a MATLAB `meshgrid`.
`ZI = griddata(mat,strike,impvol,XI,YI);`	Perform the interpolation.
`mesh(XI,YI,ZI);`	Produce a 3D-plot.
`hold on`	Print the next plot over this one.
`plot3(mat, strike, impvol,'o');`	Plot the actual data points.
`set(gca,'YDir','reverse');`	Change strike axis for more natural view.
`view([-74.5 34]);`	Move to a nicer viewpoint.

This code produces the following plot:



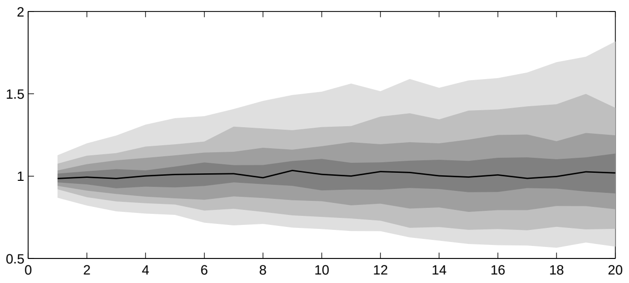
0.24
0.22
0.2
0.18
0.16
2000
1800
1600
1400
1200
1000
0
0.2
0.4
0.6
0.8
1

Quantiles of a distribution of sample paths

The following plot is inspired by the inflation report of the Bank of England,[1] which publishes not only the median inflation forecast, but also its quantiles.

```
N=200; T=20                                Setup: N sample paths; T time periods
r=randn(T,N)*0.1;                          Simulate some log-returns ...
p=exp(cumsum(r));                          ... and obtain some prices.
pqua=quantile(p.',.1:.1:.9);               Deciles of price distribution at every t.
for q=1:4                                  Loop over deciles 1-4
  mycol=(1-q/8)*[1 1 1];                   Define greys as RGB values.
  fill([1:T T:-1:1],[pqua(q,:) pqua(10-q,T:-1:1)],mycol,'EdgeColor','none');
  hold on                                  Previous line plots grey area for 1st-9th decile,
end                                        then 2nd-8th and so on
plot(1:20,pqua(5,:),'k')                   Add median (5th decile) as black line
```

This code produces the following plot:



Function-driven plots

```
myfunction=@(x)0.1*x.^3-x.^2+x;     Dotted operators: vector-compatible function
fplot(myfunction,[-5 10])
```

Saving and exporting a plot

You can use File/Save As ... in the window of the plot and select the appropriate file format to save your plots. MATLAB sometimes crops figures, so immediately check your output. It is much better to include the following lines of code to export your figures:

```
h=figure('Visible','off','PaperUnits','centimeters','PaperSize',[20 15],...
         'PaperPositionMode','manual','PaperPosition',[0 0 20 15]);
```

Add your plot commands here.

```
saveas(h,filemame,format);
```

[1]See `http://www.bankofengland.co.uk/publications/Pages/inflationreport/`.

The first line creates a new (empty) plot. `Visible off` tells MATLAB not to show it on the screen, `PaperSize` is the width and and height in centimeters, the remaining commands fix the position of the graph in the center. Finally, `filename` is a variable that contains your file name and format is one of the following formats: `eps, pdf, jpg, png`.

Annotating plots

To annotate plots with LaTeX commands, set LaTeX as *string interpreter*. This works with `title, legend, xlabel, ylabel`. Example:

```
title('$\sqrt{x}$','Interpreter','latex')
```

Changing a plot

Use the command `plottools` or click on the □ symbol in the figure window/toolstrip to change the design of a plot. You can turn on/off individual data sets, move the location of the caption, change colours and line styles and do many more things.

D.3. MATLAB table output

There is no command for table output in MATLAB. An improvised way to produce tables is to create a matrix by horizontal concatenation.

```
FirstCol=(0.1:0.1:1).'          Create some data ...
SecondCol=(100:100:1000).'
disp([FirstCol SecondCol]);     Improvised table.
format bank                     Better scaling of numbers ...
disp([FirstCol SecondCol]);     Better table.
```

For better tables use `sprintf(fomatting_string, variables)`. For details how to define the formatting string, use `doc sprintf`.

```
sprintf('%5.2f | %d \n',[FirstCol SecondCol]')
```

TIP: For really nice tables with many options, install the `datatable` package.

D.4. From MATLAB to LaTeX– creating a research workflow

A large MATLAB project may easily contain hundreds of files which have different uses. They also have different *scopes*: some may be of very general use, some may be only useful for one special case. Like adopting naming convention, it has proved very useful to adhere to a *location convention*, i.e. to organize files in different places to avoid confusion. The guiding principle should be to avoid multiple copies of one file. These are a special source of errors, as it will sooner or later happen that one of the copies is changed (e.g. to correct an error), while the other ones are not.

One possible way to organize your files is the following:

Project directory

It is obvious that when starting a project, you want to put everything in one place – the project directory. This is the place, where project-specific code should rest. However, there are three exceptions, as described in the following:

Your personal toolbox

MATLAB scripts can only "see" files that are in the same folder or on the MATLAB path (more below). It is a common mistake to therefore copy functions of general use into each project folder. Every function that is used more than once should be stored in a central place to ensure consistency. These can be

- Open source toolboxes downloaded from the web
- User-defined functions that are of general use

Create a folder called `MATLABtoolbox` that is located in a convenient place. Using the command File/Set Path ... adds this folder to the list of places where MATLAB looks for functions/m-files. Now you can conveniently call your standard functions from every script and you have a well-organized personal library.

Note: you may want to organize your personal library in sub-folders, like for example `MATLABtoolbox/Econometrics` or `MATLABtoolbox/Options`. If you do so, use Add With Subfolders ... in the Set Path dialogue. Nevertheless, if you decide to create a subfolder later, you have to add it manually using File/Set Path ....

In any case, only tested and documented functions should make it into your toolbox. If you work a lot with MATLAB, you may want to setup a `MATLABtoolboxDev` as well, where you store the not fully-tested versions. If you do so, take care that `MATLABtoolbox` is on the MATLAB path before `MATLABtoolboxDev` to ensure that MATLAB searches first in the toolbox and then in the developer folder.

Data directory

Following the same logic as for the personal toolbox, it is also a good idea to put all data into a folder and add it to the MATLAB path. These two tips make it possible to create different versions of a program by simply duplicating its folder, while avoiding inconsistencies in library functions or the data set.

Work with run files

A run file is a small MATLAB program that contains only initialization code (mostly loading a specific data set and choosing model parameters) and the calls the main program. When working on a research project, you will probably want to test different versions of your model with different input parameters and possibly different data sets. Simply changing the parameters in the code is not a good idea: you might loose this information and get stuck with perfect results that "only" lack the data generating process. An alternative is to duplicate the whole project folder before changing the parameters. This ensures that nothing is lost, but may lead to inconsistencies as you change code. Best is to put all code in a directory and add this to the MATLAB path (like the personal library). Then create one folder per run containing the run file and all results.

It is also a good idea to save important parameters with the data, just in case it gets separated from its run file. This can be easily done by employing MATLAB structures:

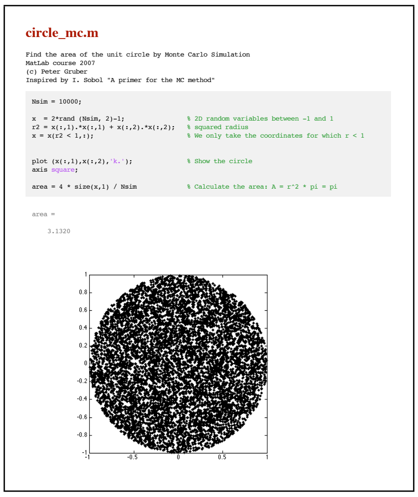
A MATLAB structure for saving data with envelope information

`mystruct.param=theta`	Saving the model parameters.
`mystruct.date=daterange`	... range of dates of the time series (if applicable)
`mystruct.model=modelname`	Information about the detailed model specification.
`mystruct.results=results`	Finally the results.
`save results.mat mystruct`	Save the whole structure.

D.5. Practitioner's tips

Use private functions to distribute code

A MATLAB project will easily consist of one main program and several functions. Distributing or printing such a number of files can become cumbersome, especially if the recipient should only run the code and will not participate in the development. In this case, private functions can provide an interesting solution. Rewrite the main program as function without arguments and add all further functions to the file.



circle_mc.m

```
Find the area of the unit circle by Monte Carlo Simulation
MatLab course 2007
(c) Peter Gruber
Inspired by I. Sobol "A primer for the MC method"
```

```
Nsim = 10000;

x  = 2*rand (Nsim, 2)-1;                 % 2D random variables between -1 and 1
r2 = x(:,1).*x(:,1) + x(:,2).*x(:,2);   % squared radius
x = x(r2 < 1,:);                         % We only take the coordinates for which r < 1


plot (x(:,1),x(:,2),'k.');               % Show the circle
axis square;

area = 4 * size(x,1) / Nsim              % Calculate the area: A = r^2 * pi = pi
```

```
area =

    3.1320
```

Figure D.1.: Output from Publish To HTML of `circle_mc.m`

`function mainprogram`	No input and output arguments
`  < the main program >`	Just copy the main program. No changes required.
`end % of funciton`	The function `mainprogram` ends with an `end`
`%% ===== private functions =====`	
`function result=myf1(arg)`	copy here all functions from their files
`end % of funciton`	End all functions with an `end`

Use cells and code folding

Cells are very useful for structuring code. Start a cell with a double percentage sign and give it a meaningful title. If applicable, add one or two lines of comment. To hide cells that are not important to your work (e.g. data loading), you have to turn on code folding. Go to File/Preferences and select Editor/Code folding. Check the box next to Cells. Now you get a small box with a minus sign left of each cell. Click once to fold the cell, click again to unfold it.

Publish to HTML

To produce some nice output, select File/Publish To HTML. This will run the main program and produce a nice HTML file that is organized section-by-section and contains all comments, the program code and the output. A fast way to produce documentation for a program is to print the HTML document to a pdf file, see Fig. D.1.

Working with a remote system

If numerical tasks are very complex, one may want to execute them on a central system instead of blocking one's personal computer for an extended amount of time. A few things have proved useful when working with a remote system

- Regularly save intermediate results. If the system crashes, you don't have to start from scratch.
- If you use solutions such as `www.dropbox.com`, you can synchronize the program folders on your PC and on the remote system. Also synchronize the file in which the intermediate results are saved. This allows for an easy check whether the system is still running.
- Email results to yourself directly from MATLAB using the `sendmail()` command. Numerical results must be converted to strings first. You can even send attachments, for example graphs that you have saved to the hard drive. You have to set MATLAB's internet preferences to be able to use `sendmail()`.
- MATLAB can also send and retrieve larger files via its built-in `ftp()` command. Use `mget()` do retrieve files and `mput()` to send files.

D.6. An introduction to the Symbolic Mathematics Toolbox

Getting started

`sqrt(2)`	Numerical solution ...
`sqrt(sym(2))`	This is now left as a symbol.
`double(ans)`	Convert to a numerical result.
`sym(1)/sym(3)+sym(1)/sym(6)`	

Symbolic variables

`syms a x`	Define a and x as a symbolic variable for later use.
`g=sym('(x+a)/(x-a)')`	A simple expression.
`simplify(g)`	One of the most useful commands.
`g=sym('1/phi+1=phi')`	The equation of the golden ratio ...
`solve(g,'phi')`	Tell MATLAB to solve with respect to which variable
`double(ans)`	Get numerical values, if you like.

Substituting for variables

`f=x^2+2*x+2`	
`subs(f,x,a/2)`	Substitutes x with $\frac{a}{2}$.

Isolating variables

`g=('Y=c*Y+G+I-b*i-m*Y+x1*Yw')`	The IS curve. Notice we left out the `sym`.
`solve(g,'Y')`	We get a neat result for $Y = \dots$ and the multiplicator.
`solve(g,'i')`	Error message, because `i` is reserved for imaginary numbers.
`syms i`	Declare `i` as a symbol.
`solve(g,'i')`	Now it works.

Systems of equations

`f=('M=h*Y-h/b*i')`	The LM-curve
`S=solve(f,g,'Y,i')`	We solve the IS-LM model for Y and i
`S.Y`	The result is a structure. We first look at the solution for Y
`S.i`	... and then for i.

Differentiate

`g=('A*K^alpha*L^(1-alpha)')`	A Cobb-Douglas production function
`diff(g,'L')`	Differentiate with respect to L.
`simplify(ans)`	Simplify the result, which is stored in `ans`.

Taylor series

`f=exp(x)`	A symbolic function
`taylor(f,3)`	The Taylor series up to order 2.
`taylor(f,5)`	A longer Taylor polynomial.
`taylor(f,3,0.5)`	Develop the polynomial around $x_0 = 0.5$.

Integrals

`f=exp(x)`	A symbolic function
`int(f)`	Indefinite integral.
`int(f,x,0,1)`	Definite integral between zero and one.
`format long`	We want to compare this to numerical integration.
`double(ans)`	Convert to number.
`g=@(y)exp(y)`	Define a numerical function.
`quad(g,0,1)`	Compare the result to the numerical integration.

Matrix Computations

`syms t`	We create a simple matrix:
`G=[cos(t) sin(t); -sin(t) cos(t)]`	
`A=G*G`	We square it.
`simplify(A)`	Nice result.
`diff(A)`	

E. MATLAB @ UNI

E.1. Guide to the exercises

- The program must give correct results.
- Programs must not crash, even if the input makes no sense.
- The rules for good programming style must be observed, especially
  - Clearly indicate the main program.
  - Programs must have sensible documentation, including a header with all relevant information. The program should be readable.
  - Name your program/functions as specified in the problem description. All other names (e.g. of variables) must be self-explanatory.
  - The program should be efficient.
  - The solution should be a general one.
  - Use foreign code (i.e. found on the web) only if allowed and then reference it correctly.
- Unless explicitly stated, decide yourself whether to write a function or a program.
- Test your program before handing it in and document your tests in the code header.
- Choose economically sensible values for the variables that are not specified in the problem definition.
- Programming projects based on Octave/Freemat are accepted. Please make a comment referring to Octave/Freemat at the begin of the main program.
- Do not use special toolbox functions. If not stated otherwise, only the *statistics* and *optimization toolboxes* are allowed.[1]

Bonus for admitting code that does not work

Understanding what does not work is a first step to solving the problem. If you notice that some part of your solution does not work as required, make a short(!) comment explaining

[1]If your are not sure if a function is part of a certain toolbox, type `doc functionname` in MATLAB, which will open a help browser. In the top of the browser, you see a hierarchy of keywords. If this hierarchy starts with MATLAB, the function is a MATLAB core function. If it starts with (Some Name) Toolbox, it is part of a toolbox.

what does not work. You will get fewer points deducted than when simply handing in a program that does not work.

Hand in interactive exercises using a diary

You can use MATLAB's "logbook" command for interactive problem sets.

`diary ('filename')`	Specify file name and start diary. Find the file in the current directory as explained in Fig. 2.1.
`%<Your text answer>`	Answer text questions as MATLAB comment. Start each line with `%`
`diary off`	Suspend (temporarily) the diary, e.g. for experimenting.
`diary on`	Start the diary again after suspension.
`diary off`	Turn the diary off at the end to close the file.

E.2. Obtaining and installing MATLAB and its open-source alternatives

The MATLAB student license is available online and at the university bookshop for a price of ca. CHF 100. The license includes the Symbolic Mathematics Toolbox, the Statistics Toolbox and the Optimization Toolbox (and a few engineering-related toolboxes).

Octave is a program under the GNU public license that can interpret MATLAB commands. It is included in many Linux distributions. Octave is like a MATLAB that contains only a command window and requires and external editor. Free text editors with syntax highlighting are Notepad++ editor for MS Windows and TextWrangler for Mac OS X. The programs are available at:

- `www.octave.org`
- `www.notepad-plus-plus.org`
- `www.barebones.com/products/textwrangler`, you need to install the MATLAB language module from `www.esm.psu.edu/~ajm138/matlab_utils`

Unfortunately, Octave is only 95% compatible to MATLAB. In many cases, minor changes are necessary to run a MATLAB program under Octave.

Currently, Octave/Freemat are not actively supported in the course (i.e. demo programs are not checked for compatibility). Students may use Octave/Freemat at their own risk.

Freemat is yet another alternative to MATLAB, which is slightly easier to use, as it has its own editor. It is available at `freemat.sourceforge.net`. For the pros and cons of Octave, MATLAB and FreeMat, see the table below.

Julia is a recent and very fast, very powerful programming language that is similar to MATLAB. Julia is free software and available at `www.julialang.org`

MATLAB	Octave	Freemat
Commercial product	Free (GPL)	Free
User-friendly (editor ...)	Calculations only	Interface similar to MATLAB
–	95% compatible to MATLAB	> 95% compatibility
MathWorks toolboxes	Public collections of code	Public collections of code
Help function (text), online documentation, printed documentation	Help function (text), wiki, can use some MATLAB documentation	Help function (text), online documentation, can use some MATLAB documentation
Activation process	Part of Linux distributions	Download, use and share
CHF 2000+ (student: 100)	CHF 0 plus your time	CHF 0 plus your time

MATLAB @ HSG in PC lab 01-U102 features 25 licenses of MATLAB and the following toolboxes: Symbolic Mathematics, Statistics, Optimization, Financial and Financial Derivatives as well as James P. LeSage's Econometrics Toolbox. MATLAB is only available there and can be used freely by students. Check if the lab is not booked.

MATLAB @ UNIGE You can buy MATLAB for classroom and personal use at `http://www.mathworks.ch/academia/student_version/`

Bibliography

Abadir, Karim M., and Jan R. Magnus (2005) *Matrix Algebra (Econometric Exercises)* (Cambrindge University Press)

Brandimarte, Paolo (2006) *Numerical Methods in Finance and Economics* (Wiley and Sons)

Carr, P., and D. Madan (1999) 'Option valuation using the fast fourier transform.' *Journal of Computational Finance* 2, 61–73

Chong, Edwin K.P., and Stanilaw H. Zak (2008) *An introduction to Optimization* (Wiley)

Christoffersen, Peter, Steve Heston, and Kris Jacobs (2009) 'The shape and term structure of the index option smirk: Why multifactor stochastic volatility models work so well.' *Management Science* 55, 1914–1932

Creel, Michael (2006) *Econometrics* (Online, `http://pareto.uab.es/mcreel/Econometrics/`)

Fang, Fang, and Kees Oosterlee (2008) 'A novel pricing method for european options based on fourier-cosine expansions.' *SIAM Journal on Scientific Computing* 31(2), 826–848

Gentle, James E. (2007) *Matrix Algebra* (Springer)

Gilli, Manfred, Dietmar Maringer, and Enrico Schumann (2011) *Numerical Methods and Optimization in Finance* (Academic Press, New York.)

Glasserman, Paul (2003) *Monte Carlo Methods in Financial Engineering* (Springer)

Greene, W. H. (2002) *Econometric Analysis* (Prentice Hall; 5th edition)

Gut, Allan (2005) *Probability: A Graduate Course* (Springer)

Hanselman, D., and B. Littlefield (2005) *Mastering Matlab 7* (Pearson Education International)

Härdle, W., and L. Simar (2003) *Applied Multivariate Statistics* (Springer)

Judd, Kenneth L. (1998) *Numerical Methods in Economics* (MIT Press)

Kerrighan, Brian W., and P. J. Plauger (1974) *The elements of programming style* (McGraw-Hill)

Kharab, A., and R. B. Guenther (2006) *Introduction to numerical methods – A MATLAB approach* (Chapman & Hall)

Kuniciky, David C. (2004) *Matlab Programming* (Pearson Prentice Hall)

LeSage, James P. (1999) *Applied Econometrics using MATLAB* (Online, `www.spatial-econometrics.com`)

Lord, Roger, and Christian Kahl (2008) 'Complex logarithms in heston-like models.' *Working paper February 21st, 2008*

Mathews, John D., and Kurtis D. Fink (2004) *Numerical Methods using Matlab* (Pearson Education, also available at `http://math.fullerton.edu/mathews/n2003/NumericalUndergradMod.html`)

Metropolis, M., and S. Ulam (1949) 'The monte carlo method.' *Journal of the American Statistical Association* 49, 335–341

Moler, Cleve (2004) *Numerical Computing with Matlab* (SIAM (also available at `http://www.mathworks.com/moler/`))

Moore, Gordon E. (1965) 'Cramming more components onto integrated circuits.' *Electronics Magazine*

Opfer, Gerhard (2004) *Numerische Mathematik für Anfänger* (Vieweg)

Pedersen, M. S., and K. B. Petersen (2007) *The Matrix Cookbook* (`http://matrixcookbook.com/`)

Simon, C. P., and L. Blume (1994) *Mathematics for economists* (Norton)

Stanoyevitch, Alexander (2005) *Introduction to MATLAB with Numerical Preliminaries* (Wile)

Verbeek, M. (2004) *Modern Econometrics* (Wiley)

Wooldridge, J. (2005) *Introductory Econometrics: A Modern Approach* (South-Western College Pub; 3 edition)

Final remark:

Writing a program is
half science and half handicraft,
so both is useful:
thinking *and* training.