Technical Report

A Specification Test for Nonparametric Instrumental Variable Regression

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This technical report contains the proofs of the technical Lemmas B.1-B.8, C.1-C.4, and D.1-D.3 in the paper entitled "A Specification Test for Nonparametric Instrumental Variable Regression" and written by P. Gagliardini and O. Scaillet. Equations labelled as (n) refer to the paper, and Equations labelled as (TR.n) refer to the technical report. To simplify the proofs, we adopt a product kernel in the estimation of the density of (Y, X, Z). We use the generic notation K for both the 3-dimensional product kernel and each of its components.

1 Proof of Lemma B.1

The result follows from (see decomposition (12)):

$$\begin{aligned}
|\xi_{1,T}| \quad &\leq \quad \max_{t\in\mathcal{T}_*}\left|\frac{(Th_T)^2\Omega_t}{(\sum_j K_{jt})^2}\right| \frac{K(0)^2}{(Th_T)^2}\frac{1}{T}\sum_t \Big(|U_t|^2 + |\mathcal{B}_T(X_t)|^2 + |\mathcal{E}_T\left(X_t\right)|^2 \\
&\quad +2\left|U_t\right|\left|\mathcal{B}_T(X_t)\right| + 2\left|U_t\right|\left|\mathcal{E}_T\left(X_t\right)\right| + 2\left|\mathcal{B}_T(X_t)\right|\left|\mathcal{E}_T\left(X_t\right)\right|\Big),
\end{aligned}$$

and $\max_{t\in\mathcal{T}_*}\left|\frac{(Th_T)^2\Omega_t}{(\sum_j K_{jt})^2}\right| = O_p(1)$, $\frac{1}{T}\sum_t |U_t|^2 = O_p(1)$, $\frac{1}{T}\sum_t |\mathcal{B}_T(X_t)|^2 = o_p(1)$, $\frac{1}{T}\sum_t \left|\hat{\varphi}(X_t) - \varphi_{\lambda_T}(X_t)\right|^2 = o_p(1)$ and the Cauchy-Schwartz inequality (Assumptions A.1-A.4, A.5 (i), 3).

2 Proof of Lemma B.2

We get from decomposition (12):

$$\begin{aligned}
\xi_{3,T} \quad = \quad & \frac{1}{T}\sum_t\sum_{s\neq t}\frac{\Omega_t K(0)}{(\sum_j K_{jt})^2}U_tU_sK_{st}I_t + \frac{1}{T}\sum_t\sum_{s\neq t}\frac{\Omega_t K(0)}{(\sum_j K_{jt})^2}\mathcal{B}_T(X_t)\mathcal{B}_T(X_s)K_{st}I_t \\
& +\frac{1}{T}\sum_t\sum_{s\neq t}\frac{\Omega_t K(0)}{(\sum_j K_{jt})^2}\mathcal{E}_T\left(X_t\right)\mathcal{E}_T\left(X_s\right)K_{st}I_t - 2\frac{1}{T}\sum_t\sum_{s\neq t}\frac{\Omega_t K(0)}{(\sum_j K_{jt})^2}U_t\mathcal{B}_T(X_s)K_{st}I_t \\
& -2\frac{1}{T}\sum_t\sum_{s\neq t}\frac{\Omega_t K(0)}{(\sum_j K_{jt})^2}U_t\mathcal{E}_T(X_s)K_{st}I_t + 2\frac{1}{T}\sum_t\sum_{s\neq t}\frac{\Omega_t K(0)}{(\sum_j K_{jt})^2}\mathcal{B}_T(X_t)\mathcal{E}_T(X_s)K_{st}I_t \\
=: \quad & \xi_{31,T} + \xi_{32,T} + \xi_{33,T} - 2\xi_{34,T} - 2\xi_{35,T} - 2\xi_{36,T}.
\end{aligned}$$

We consider in details the first three terms (the bounds for the remaining terms are similar).

The term $\xi_{31,T}$ corresponds to statistic $\hat{T}_3^{(1)}$ of TK, p. 2082 (multiplied by T^{-1} and for a given weighting function). Along the lines of Lemma A.4 in TK, we have $\xi_{31,T} =$

$O_p\left(\frac{1}{(Th_T)^{3/2}}\right)O_p\left(\sup_{z\in S_*}\left|\hat{f}(z)^{-1}-f(z)^{-1}\right|\right)$. From the uniform convergence of the kernel density estimator (Assumptions A.1, A.3, A.4) and $h_T=\bar{c}T^{-\bar{\eta}}$ with $\bar{\eta}<2/3$ (Assumption 3), we get $\xi_{31,T}=o_p((Th_T^{1/2})^{-1})$.

Let us now consider the second term, $\xi_{32,T}$. Define $\eta_s:=\mathcal{B}_T\left(X_s\right)-E\left[\mathcal{B}_T\left(X_s\right)|Z_s\right]$ and $b_s:=E\left[\mathcal{B}_T\left(X_s\right)|Z_s\right]=\left(A\mathcal{B}_T\right)(Z_s)$. Then:

$$\begin{aligned}\xi_{32,T} &= \frac{1}{T}\sum_t\sum_{s\neq t}\frac{\Omega_tK(0)}{(\sum_j K_{jt})^2}b_tb_sK_{st}I_t+\frac{1}{T}\sum_t\sum_{s\neq t}\frac{\Omega_tK(0)}{(\sum_j K_{jt})^2}\eta_t\eta_sK_{st}I_t\\ &\quad+\frac{1}{T}\sum_t\sum_{s\neq t}\frac{\Omega_tK(0)}{(\sum_j K_{jt})^2}b_t\eta_sK_{st}I_t+\frac{1}{T}\sum_t\sum_{s\neq t}\frac{\Omega_tK(0)}{(\sum_j K_{jt})^2}\eta_tb_sK_{st}I_t\\ &=: \xi_{321,T}+\xi_{322,T}+\xi_{323,T}+\xi_{324,T}.\end{aligned}$$

By the uniform convergence of the kernel density estimator, the dominant term in $\xi_{321,T}$ is

$$\xi_{3211,T}=\frac{1}{T^3h_T^2}\sum_t\sum_{s\neq t}\frac{\Omega_tK(0)}{f(Z_t)^2}b_tb_sK_{st}I_t=\frac{K(0)}{T^2h_T}\sum_t\frac{\Omega_t}{f(Z_t)^2}I_tb_t\left(\frac{1}{Th_T}\sum_{s\neq t}b_sK_{st}\right).$$

Using that $E\left[\frac{\Omega_t}{f(Z_t)^2}I_tb_t\left(\frac{1}{Th_T}\sum_{s\neq t}b_sK_{st}\right)\right]=E\left[\frac{\Omega_t}{f(Z_t)}I_tb_t^2\right](1+o(1))=O\left(E\left[\Omega_tI_t\left(A\mathcal{B}_T\right)(Z_t)^2\right]\right)$, $E\left[\Omega_tI_t\left(A\mathcal{B}_T\right)(Z_t)^2\right]=Q_{\lambda_T}=O\left(\lambda_T^{1+\beta}\right)$ (see Appendix A.2.3), and Assumption 3, it follows that $\xi_{321,T}=O_p\left(\frac{1}{Th_T}Q_{\lambda_T}\right)=o_p\left(\left(Th_T^{1/2}\right)^{-1}\right)$. The dominant term in $\xi_{322,T}$ is

$$\xi_{3221,T}=\frac{1}{T^3h_T^2}\sum_t\sum_{s\neq t}\frac{\Omega_tK(0)}{f(Z_t)^2}\eta_t\eta_sK_{st}I_t=\frac{1}{T^3h_T^2}\sum_t\sum_{s>t}a_{ts}\eta_t\eta_s=:\frac{1}{T^3h_T^2}J_{3221,T},$$

where $a_{ts}=\frac{\Omega_tK(0)}{f(Z_t)^2}K_{st}I_t+\frac{\Omega_sK(0)}{f(Z_s)^2}K_{ts}I_s$. Using that $E\left[\eta_t|\mathcal{I}\right]=0$ and $E\left[\eta_t\eta_s|\mathcal{I}\right]=0$ for $t\neq s$, from the independence of the observations, we have:

$$E\left[J_{3221,T}^2\right]=\sum_t\sum_{s>t}E\left[a_{ts}^2\eta_t^2\eta_s^2\right]=\sum_t\sum_{s>t}E\left[a_{ts}^2\Gamma(Z_t)\Gamma(Z_s)\right],$$

where $\Gamma(Z_t):=E\left[\eta_t^2|Z_t\right]=V\left[\mathcal{B}_T\left(X_t\right)|Z_t\right]$, and the cross-terms vanish because of the conditional independence property of the η_t variables. Then, we get $E\left[J_{3221,T}^2\right]=O\left(T^2h_T\right)$ and thus $\xi_{322,T}=O_p\left(\frac{1}{T^2h_T^{3/2}}E\left[\eta_t^2\right]\right)=o_p\left(\left(Th_T^{1/2}\right)^{-1}\right)$. The argument is similar for $\xi_{323,T}$ and $\xi_{324,T}$, and we deduce $\xi_{32,T}=o_p\left(\left(Th_T^{1/2}\right)^{-1}\right)$.

Let us finally consider the third term, $\xi_{33,T}$. We have

$$|\xi_{33,T}| \leq \max_{t\in\mathcal{T}_*} \left| \frac{(Th_T)^2\Omega_t}{(\sum_j K_{jt})^2} \right| \frac{K(0)}{Th_T} \frac{1}{T^2 h_T} \sum_t \sum_{s\neq t} |\mathcal{E}_T(X_t)| \, |\mathcal{E}_T(X_s)| \, K_{st} I_t.$$

Applying the Cauchy-Schwarz inequality twice, we deduce:

$$\frac{1}{T^2 h_T} \sum_t \sum_{s\neq t} |\mathcal{E}_T(X_t)\mathcal{E}_T(X_s)| \, K_{st} I_t \leq \frac{1}{T} \sum_t |\mathcal{E}_T(X_t)|^2 \sqrt{\frac{1}{T^2 h_T^2} \sum_t \sum_{s\neq t} K_{st}^2 I_t}.$$

From $E\left[\frac{1}{T^2 h_T^2} \sum_t \sum_{s\neq t} K_{st}^2 I_t\right] = O(h_T^{-1})$ we get $\xi_{33,T} = O_p\left(\frac{1}{Th_T^{1/2}} \frac{1}{h_T} \left(\frac{1}{T} \sum_t \left|\hat{\varphi}(X_t) - \varphi_{\lambda_T}(X_t)\right|^2\right)\right)$.
It follows $\xi_{33,T} = o_p((Th_T^{1/2})^{-1})$ from Assumptions A.5 (i) and 3.

3 Proof of Lemma B.3

Define $\eta_s := \mathcal{B}_T(X_s) - E\left[\mathcal{B}_T(X_s)\,|Z_s\right]$ and $b_s := E\left[\mathcal{B}_T(X_s)\,|Z_s\right] = (A\mathcal{B}_T)(Z_s)$. Split

$$\mathcal{K}_T(\mathcal{B}_T(X), \mathcal{B}_T(X)) = \mathcal{K}_T(b, b) + 2\mathcal{K}_T(b, \eta) + \mathcal{K}_T(\eta, \eta) =: J_{11,T} + J_{12,T} + J_{13,T}.$$

Then, term $J_{11,T}$ can be written as

$$\begin{aligned} J_{11,T} &= \frac{1}{T} \sum_t \frac{(Th_T)^2 \Omega_t I_t}{\left(\sum_j K_{jt}\right)^2} \frac{1}{T^2 h_T^2} \left(\sum_{s\neq t} K_{st} b_s\right)^2 - \frac{1}{T} \sum_t \frac{(Th_T)^2 \Omega_t I_t}{\left(\sum_j K_{jt}\right)^2} \frac{1}{T^2 h_T^2} \sum_{s\neq t} K_{st}^2 b_s^2 \\ &=: J_{111,T} - J_{112,T}, \end{aligned}$$

where $J_{111,T}$ is the dominant term. Using

$$\begin{aligned} J_{111,T} &= \frac{1}{T} \sum_t \frac{\Omega_t I_t}{f(Z_t)^2} \frac{1}{T^2 h_T^2} \left(\sum_{s\neq t} K_{st} b_s\right)^2 \\ &\quad + \frac{1}{T} \sum_t \left[\frac{(Th_T)^2}{\left(\sum_j K_{jt}\right)^2} - \frac{1}{f(Z_t)^2}\right] \Omega_t I_t \frac{1}{T^2 h_T^2} \left(\sum_{s\neq t} K_{st} b_s\right)^2, \end{aligned}$$

$$E\left[\frac{\Omega_t I_t}{f(Z_t)^2} \frac{1}{T^2 h_T^2} \left(\sum_{s\neq t} K_{st} b_s\right)^2\right] = E\left[\Omega_t I_t \left[(A\mathcal{B}_T)(Z_t)\right]^2\right](1 + o(1)),$$

$\inf_{z\in S_*} \frac{\Omega_0(z)}{f(z)^2} > 0$, $\sup_{t\in\mathcal{T}^*} \left|\frac{(Th_T)^2}{\left(\sum_j K_{jt}\right)^2} - \frac{1}{f(Z_t)^2}\right| = o_p(1)$, we deduce $J_{111,T} = Q_{\lambda_T}(1 + o_p(1))$.

Terms $J_{12,T}$ and $J_{13,T}$ can be analyzed similarly, and we consider only $J_{13,T}$ in details. Write

$$\begin{aligned}
J_{13,T} &= \frac{1}{T}\sum_t \frac{\Omega_t I_t}{f\left(Z_t\right)^2}\frac{1}{T^2 h_T^2}\sum_{s\neq t}\sum_{u\neq t,s} K_{st}K_{ut}\eta_s\eta_u \\
&\quad + \frac{1}{T}\sum_t \left[\frac{\left(Th_T\right)^2}{\left(\sum_j K_{jt}\right)^2} - \frac{1}{f\left(Z_t\right)^2}\right]\Omega_t I_t \frac{1}{T^2 h_T^2}\sum_{s\neq t}\sum_{u\neq t,s} K_{st}K_{ut}\eta_s\eta_u \\
&=: J_{131,T} + J_{132,T}.
\end{aligned}$$

Note that $E\left[\eta_s|\mathcal{I}\right] = 0$ and $E\left[\eta_s\eta_u|\mathcal{I}\right] = 0$ for $s \neq u$, from the independence of the observations. Along the lines of Lemma A.7 in TK, using Assumptions A.1-A.4 and 3 we can prove that $J_{132,T} = o_p((Th_T^{1/2})^{-1})$. Moreover, we have $J_{131,T} = \dfrac{1}{T}\dfrac{1}{T^2 h_T^2}J_{1,T}^*$, where $J_{1,T}^* = \sum_s \sum_{u>s} c_{su}\eta_s\eta_u$ and $c_{su} := 2\displaystyle\sum_{t\neq s,u}\frac{\Omega_t I_t}{f\left(Z_t\right)^2}K_{st}K_{ut}$. Then, we get

$$E\left[J_{1,T}^{*2}\right] = \sum_s\sum_{u>s} E\left[c_{su}^2\eta_s^2\eta_u^2\right] = \sum_s\sum_{u>s} E\left[c_{su}^2\Gamma(Z_s)\Gamma(Z_u)\right],$$

where $\Gamma(Z_s) := E\left[\eta_s^2|Z_s\right] = V\left[\mathcal{B}_T\left(X_s\right)|Z_s\right]$, and the cross-terms vanish because of the conditional independence property of the η_s variables. To compute $E\left[c_{su}^2\Gamma(Z_s)\Gamma(Z_u)\right]$, we can use an argument similar to that in Lemma A.8 of TK, to get $E\left[c_{su}^2\Gamma(Z_s)\Gamma(Z_u)\right] = O\left(T^2 h_T^3 E\left[\dfrac{\Omega_0(Z_t)I_t}{f\left(Z_t\right)}\Gamma(Z_t)\right]^2\right)$. Using Assumptions A.1, A.3, A.4, we have $E\left[\dfrac{\Omega_0(Z_t)I_t}{f\left(Z_t\right)}\Gamma(Z_t)\right] \leq const \cdot b(\lambda_T)^2$, where $b(\lambda_T) := \langle\mathcal{B}_T, \mathcal{B}_T\rangle^{1/2} = o(1)$. Thus, we deduce that $J_{131,T} = o_p((Th_T^{1/2})^{-1})$. The conclusion follows.

4 Proof of Lemma B.4

With the notation in the proof of Lemma B.3 we have

$$\mathcal{K}_T\left(U, \mathcal{B}_T(X)\right) = \mathcal{K}_T\left(U, b\right) + \mathcal{K}_T\left(U, \eta\right) =: J_{21,T} + J_{22,T}.$$

Let us first consider $J_{21,T}$. By assumptions A.1-A.4, 3, and an argument similar to Lemma A.7 of TK, we have

$$\begin{aligned}
J_{21,T} &= \frac{1}{T}\frac{1}{T^2 h_T^2}\sum_t \frac{\Omega_t I_t}{f(Z_t)^2}\sum_{s\neq t}\sum_{u\neq t,s} K_{st}K_{ut}U_s b_u + o_p((Th_T^{1/2})^{-1}) \\
&= \frac{1}{T^3 h_T^2}\sum_s a_s U_s + o_p((Th_T^{1/2})^{-1}) \qquad =: \quad \frac{1}{T^3 h_T^2}J_{2,T}^* + o_p((Th_T^{1/2})^{-1}),
\end{aligned}$$

where $a_s = \sum_{t\neq s} \frac{\Omega_t I_t}{f(Z_t)^2} K_{st} \sum_{u\neq t,s} K_{ut} b_u$. From the independence of the observations and the conditional moment restriction, $E\left[\left(J_{2,T}^*\right)^2\right] = \sum_s E\left[a_s^2 U_s^2\right] = \sum_s E\left[a_s^2 V_0(Z_s)\right]$. To compute the expectation $E\left[a_s^2 V_0(Z_s)\right]$, we use

$$\begin{aligned}
E\left[a_s^2 V_0(Z_s)\right] &= \sum_{t\neq s} E\left[\frac{\Omega_t^2 I_t}{f(Z_t)^4} V_0(Z_s) K_{st}^2 \left(\sum_{u\neq t,s} K_{ut} b_u\right)^2\right] \\
&+ \sum_{t\neq s} \sum_{i\neq t,s} E\left[\frac{\Omega_t I_t}{f(Z_t)^2} \frac{\Omega_i I_i}{f(Z_i)^2} V_0(Z_s) K_{st} K_{si} \left(\sum_{u\neq t,s} K_{ut} b_u\right)\left(\sum_{m\neq i,s} K_{mi} b_m\right)\right],
\end{aligned}$$

where the second term is the dominant one. Moreover, for $t \neq s \neq i \neq u \neq m$,

$$E\left[V_0(Z_s) K_{st} K_{si} K_{ut} K_{mi} b_u b_m | Z_t, Z_i\right] = O_p\left(h_T^3 V_0(Z_t) f\left(Z_t\right)^2 f\left(Z_i\right) K * K\left(\frac{Z_i - Z_t}{h_T}\right) b_t b_i\right).$$

Thus we get $E\left[a_s^2 V_0(Z_s)\right] = O\left(T^4 h_T^4 E\left[\Omega_t I_t b_t^2\right]\right)$. We deduce

$$J_{21,T} = O_p\left(\frac{1}{\sqrt{T}} E\left[\Omega_t I_t \left[\left(A\mathcal{B}_T\right)\left(Z_t\right)\right]^2\right]^{1/2}\right) + o_p((T h_T^{1/2})^{-1}).$$

The second term $J_{22,T}$ can be analyzed along the same lines as term $J_{13,T}$ in the proof of Lemma B.3, using $E\left[\eta_u | \mathcal{I}, W_s\right] = 0$, for $u \neq s$, and $E\left(\eta_u^2\right) = o(1)$. Hence $J_{22,T} = o_p((T h_T^{1/2})^{-1})$, and the conclusion follows.

5 Proof of Lemma B.5

We give details for the bounds of terms $\mathcal{K}_T\left(\mathcal{E}_{T,k}(X), \mathcal{E}_{T,k}(X)\right)$, $k = 1, 2$. The term $\mathcal{K}_T\left(\mathcal{E}_{T,1}(X), \mathcal{E}_{T,2}(X)\right)$ is bounded similarly.

5.1 Bound of $\mathcal{K}_T\left(\mathcal{E}_{T,1}(X), \mathcal{E}_{T,1}(X)\right)$

Write:

$$\begin{aligned}
\hat{\psi}(z) &= \frac{\frac{1}{T h_T} \sum_n U_n K\left(\frac{Z_n - z}{h_T}\right)}{f(z)} + \frac{\frac{1}{T h_T} \sum_n G_{n,T} K\left(\frac{Z_n - z}{h_T}\right)}{f(z)} \\
&=: \frac{1}{T} \sum_n U_n \omega_n(z) + \frac{1}{T} \sum_n G_{n,T} \omega_n(z),
\end{aligned}$$

where $G_{n,T} := \int \left[\varphi_0(X_n) - \varphi_0(X_n + uh_T)\right] K(u)du$. Then we have $\mathcal{E}_{T,1}(X_s) = \frac{1}{T}\sum_n U_n \Psi_{sn} + \frac{1}{T}\sum_n G_{n,T}\Psi_{sn}$, where $\Psi_{sn} := \left((\lambda_T + A^*A)^{-1} A^*\omega_n\right)(X_s)$. We get

$$\begin{aligned}
\mathcal{K}_T\left(\mathcal{E}_{T,1}(X), \mathcal{E}_{T,1}(X)\right) &= \frac{1}{T^3}\sum_t \frac{\Omega_t I_t}{\left(\sum_j K_{jt}\right)^2}\sum_n\sum_m U_n U_m \left(\sum_{s\neq t}\sum_{u\neq s,t} K_{st}K_{ut}\Psi_{sn}\Psi_{um}\right) \\
&\quad + \frac{1}{T^3}\sum_t \frac{\Omega_t I_t}{\left(\sum_j K_{jt}\right)^2}\sum_n\sum_m G_{n,T}G_{m,T}\left(\sum_{s\neq t}\sum_{u\neq s,t} K_{st}K_{ut}\Psi_{sn}\Psi_{um}\right) \\
&\quad + 2\frac{1}{T^3}\sum_t \frac{\Omega_t I_t}{\left(\sum_j K_{jt}\right)^2}\sum_n\sum_m U_n G_{m,T}\left(\sum_{s\neq t}\sum_{u\neq s,t} K_{st}K_{ut}\Psi_{sn}\Psi_{um}\right) \\
&=: J_{31,T} + J_{32,T} + 2J_{33,T}.
\end{aligned}$$

Let us first consider term $J_{31,T}$. Define $Q_{sn} := E\left[\Psi_{sn}|\mathcal{I}\right] = \left(A\left(\lambda_T + A^*A\right)^{-1} A^*\omega_n\right)(Z_s)$ and $V_{sn} := \Psi_{sn} - Q_{sn}$. Then:

$$\begin{aligned}
J_{31,T} &= \frac{1}{T^3}\sum_t \frac{\Omega_t I_t}{\left(\sum_j K_{jt}\right)^2}\sum_n\sum_m U_n U_m \left(\sum_{s\neq t}\sum_{u\neq s,t} K_{st}K_{ut}Q_{sn}Q_{um}\right) \\
&\quad + \frac{1}{T^3}\sum_t \frac{\Omega_t I_t}{\left(\sum_j K_{jt}\right)^2}\sum_n\sum_m U_n U_m \left(\sum_{s\neq t}\sum_{u\neq s,t} K_{st}K_{ut}V_{sn}V_{um}\right) \\
&\quad + 2\frac{1}{T^3}\sum_t \frac{\Omega_t I_t}{\left(\sum_j K_{jt}\right)^2}\sum_n\sum_m U_n U_m \left(\sum_{s\neq t}\sum_{u\neq s,t} K_{st}K_{ut}Q_{sn}V_{um}\right) \\
&=: J_{311,T} + J_{312,T} + J_{313,T}. \qquad \text{(TR.1)}
\end{aligned}$$

We consider first term $J_{311,T}$. By the uniform convergence of the kernel density estimator and arguments similar to Lemmas A.6 and A.7 in TK, we have

$$\begin{aligned}
J_{311,T} &= \frac{1}{T^3}\sum_t H_0(Z_t)^{-1} I_t \sum_{n\neq t}\sum_{m\neq n,t} U_n U_m \left(\frac{1}{T^2 h_T^2}\sum_{s\neq t}\sum_{u\neq s,t} K_{st}K_{ut}Q_{sn}Q_{um}\right) + o_p((Th_T^{1/2})^{-1}) \\
&=: \frac{1}{T^3} J^*_{3,T} + o_p((Th_T^{1/2})^{-1}). \qquad \text{(TR.2)}
\end{aligned}$$

Term $J^*_{3,T}$ can be written as $J^*_{3,T} = \sum_n \sum_{m>n} \gamma_{nm} U_n U_m$, where

$$\gamma_{nm} := 2\sum_{t\neq n,m} H_0(Z_t)^{-1} I_t \left(\frac{1}{T^2 h_T^2}\sum_{s\neq t}\sum_{u\neq s,t} K_{st}K_{ut}Q_{sn}Q_{um}\right).$$

By using that variables U_n and U_m are uncorrelated conditional on $\mathcal{I}$, we have

$$E\left[J_{3,T}^{*2}\right] = \sum_n \sum_{m>n} E\left[\gamma_{nm}^2 U_n^2 U_m^2\right] = \sum_n \sum_{m>n} E\left[\gamma_{nm}^2 V_0(Z_n) V_0(Z_m)\right].$$

To compute the expectation, we use an argument similar to Lemma A.8 in TK. To simplify let $\Omega_0(z) = V_0(z)^{-1} = 1$. Then, $E\left[\gamma_{nm}^2\right] = O\left(\sum_{t=1, t\neq n,m}^{T} \sum_{i=1, i\neq n,m,t}^{T} R_{ti}\right)$, where

$$R_{ti} := E\left[I_t I_i \left(\frac{1}{T^2 h_T^2} \sum_{s\neq t} \sum_{u\neq s,t} K_{st} K_{ut} Q_{sn} Q_{um}\right)\left(\frac{1}{T^2 h_T^2} \sum_{p\neq i} \sum_{q\neq s,i} K_{pi} K_{qi} Q_{pn} Q_{qm}\right)\right].$$

Developing the sums, using $\frac{1}{h_T} E\left[K_{st} Q_{sn} | Z_t, Z_n\right] = O_p\left(f\left(Z_t\right) Q_{tn}\right)$ for $s \neq t, n$, and the independence of observations, we get

$$R_{ti} = O\left(E\left[I_t I_i Q_{tn} Q_{tm} Q_{in} Q_{im}\right]\right) = O\left(E\left[I_t I_i E\left[Q_{tn} Q_{in} | Z_t, Z_i\right]^2\right]\right). \tag{TR.3}$$

To compute expectations involving Q_{tn}, we use a development of $\left(\lambda_T + A^* A\right)^{-1} A^* \omega_n$ w.r.t. the basis of eigenfunctions ϕ_j of $A^* A$ to eigenvalues ν_j:

$$A\left(\lambda_T + A^* A\right)^{-1} A^* \omega_n = \sum_{j=1}^{\infty} \frac{1}{\lambda_T + \nu_j} \left\langle \phi_j, A^* \omega_n \right\rangle_H A\phi_j = \sum_{j=1}^{\infty} \frac{1}{\lambda_T + \nu_j} \left\langle A\phi_j, \omega_n \right\rangle_{L^2(F_Z)} A\phi_j.$$

Thus $Q_{tn} = \sum_{j=1}^{\infty} \frac{1}{\lambda_T + \nu_j} c_{nj} A\phi_j\left(Z_t\right)$ where

$$c_{nj} := \left\langle A\phi_j, \omega_n \right\rangle_{L^2(\mathcal{Z})} = \frac{1}{h_T} \int A\phi_j(z) K\left(\frac{Z_n - z}{h_T}\right) dz = \int A\phi_j(Z_n - h_T u) K\left(u\right) du.$$

Then

$$E\left[Q_{tn} Q_{in} \mid Z_t, Z_i\right] = \sum_{j=1}^{\infty} \sum_{l=1}^{\infty} \frac{1}{\lambda_T + \nu_j} \frac{1}{\lambda_T + \nu_l} E\left[c_{nj} c_{nl}\right] A\phi_j\left(Z_t\right) A\phi_l\left(Z_i\right). \tag{TR.4}$$

From the orthogonality of the eigenfunctions, and the independence of the observations, we get $E\left[E\left[Q_{tn} Q_{in} | Z_t, Z_i\right]^2\right] = \sum_{j=1}^{\infty} \sum_{l=1}^{\infty} \frac{\nu_j}{\left(\lambda_T + \nu_j\right)^2} \frac{\nu_l}{\left(\lambda_T + \nu_l\right)^2} E\left[c_{nj} c_{nl}\right]^2$, for $t \neq i$. Moreover, from Assumptions A.4 (i)-(ii) and A.7 (ii) we have

$$\begin{aligned} E\left[c_{nj} c_{nl}\right] &= E\left[A\phi_j(Z_n) A\phi_l(Z_n)\right] + O(h_T^2)\left(E\left[A\phi_j(Z_n)^2\right]^{1/2} + E\left[A\phi_l(Z_n)^2\right]^{1/2}\right) + O(h_T^4) \\ &= \nu_j \delta_{jl} + O(h_T^2)\left(\sqrt{\nu_j} + \sqrt{\nu_l}\right) + O(h_T^4), \end{aligned} \tag{TR.5}$$

uniformly in j, l, where δ_{jl} is the Kronecker delta. Thus we get

$$\begin{aligned} R_{ti} &= O\left(\sum_{j=1}^{\infty}\frac{\nu_j^4}{(\lambda_T+\nu_j)^4}+h_T^4\sum_{j=1}^{\infty}\frac{\nu_j^2}{(\lambda_T+\nu_j)^2}\sum_{l=1}^{\infty}\frac{\nu_l}{(\lambda_T+\nu_l)^2}+h_T^8\left(\sum_{j=1}^{\infty}\frac{\nu_j}{(\lambda_T+\nu_j)^2}\right)^2\right) \\ =: &\quad O\left(S\left(\lambda_T\right)\right). \end{aligned}$$

Thus, $E\left[J_{3,T}^{*2}\right] = O\left(T(T-1)(T-2)(T-3)S\left(\lambda_T\right)\right)$, which implies $Th_T^{1/2}J_{311,T} = O_p\left(\sqrt{h_T S\left(\lambda_T\right)}\right)+o_p(1)$. Using that $\sum_{j=1}^{\infty}\frac{\nu_j}{(\lambda_T+\nu_j)^2} = O\left(\frac{1}{\lambda_T}\right)$ and $\sum_{j=1}^{\infty}\frac{\nu_j^4}{(\lambda_T+\nu_j)^4} \leq \sum_{j=1}^{\infty}\frac{\nu_j^2}{(\lambda_T+\nu_j)^2} \leq \sum_{j=1}^{\infty}\frac{\nu_j}{\lambda_T+\nu_j} = O\left(\log(1/\lambda_T)\right)$ under Assumption B.7 (see GS, proof of Lemma A.6), we get $S\left(\lambda_T\right) = O\left(\log(1/\lambda_T)\right)+O\left(h_T^4\frac{1}{\lambda_T}\log(1/\lambda_T)\right)+O\left(h_T^8\frac{1}{\lambda_T^2}\right)$. Then, $S\left(\lambda_T\right) = O(\log(1/\lambda_T))$ follows from $\lambda_T = cT^{-\gamma}$ with $\gamma < 4\bar{\eta}$ (Assumption 4), and we get $J_{311,T} = o_p(1/(Th_T^{1/2}))$.

Let us now consider $J_{312,T}$ in (TR.1). By the uniform convergence of the kernel density estimator and arguments similar to Lemmas A.6 and A.7 in TK, we have

$$\begin{aligned} J_{312,T} &= \frac{1}{T^5h_T^2}\sum_t H_0(Z_t)^{-1}I_t\sum_{n\neq t}\sum_{m\neq n,t}\sum_{s\neq t}\sum_{u\neq s,t}K_{st}K_{ut}U_nU_mV_{sn}V_{um}+o_p((Th_T^{1/2})^{-1}) \\ &= \frac{1}{T^4}\sum_n\sum_{m\neq n}\sum_s\sum_{u\neq s}\chi_{nmsu}U_nU_mV_{sn}V_{um}+o_p((Th_T^{1/2})^{-1}) =: J_{312,T}^*+o_p((Th_T^{1/2})^{-1}), \end{aligned}$$

where $\chi_{nmsu} := \frac{1}{Th_T^2}\sum_{t\neq n,m,s,u}H_0(Z_t)^{-1}I_tK_{st}K_{ut}$. Using that $E\left[U_n|\mathcal{I},W_m\right]=0$ for $m\neq n$, $E\left[V_{sn}|\mathcal{I},W_u\right]=0$ for $u\neq s$, and developing the expressions of the conditional variances, we deduce that $E\left[\left(J_{312,T}^*\right)^2\right] = O\left(1/\left(T^4h_T\lambda_T^2\right)\right)$. From $\lambda_T = cT^{-\gamma}$, $\gamma<1$ (Assumption 4), it follows $J_{312,T} = o_p((Th_T^{1/2})^{-1})$. Similar arguments apply for $J_{313,T}$, and from (TR.1) we get $J_{31,T} = o_p((Th_T^{1/2})^{-1})$.

Let us now consider $J_{32,T}$. Similarly as in (TR.1) and (TR.2), we have $J_{32,T} = \frac{1}{T^3}J_{3,T}^{**}+o_p((Th_T^{1/2})^{-1})$, where $J_{3,T}^{**} = \sum_n\sum_{m\neq n}\gamma_{nm}G_{n,T}G_{m,T}$. From the above arguments we have $\gamma_{nm} = O_p\left(T\sqrt{S(\lambda_T)}\right)$ uniformly in n,m. Moreover, from Assumption A.8, $G_{n,T} = O_p(h_T^2)$ uniformly in n. Thus, $J_{32,T} = O_p(h_T^4\sqrt{S(\lambda_T)})+o_p((Th_T^{1/2})^{-1})$. Since $S(\lambda_T) = O\left(\log(1/\lambda_T)\right)$ (see above), $J_{32,T} = o_p((Th_T^{1/2})^{-1})$ follows from Assumptions 3 and 4. Similar arguments apply to $J_{33,T}$, and the proof is concluded.

5.2 Bound of $\mathcal{K}_T\left(\mathcal{E}_{T,2}(X), \mathcal{E}_{T,2}(X)\right)$

We have

$$
\begin{aligned}
\mathcal{E}_{T,2}(x) &= (\lambda_T + A^*A)^{-1}\left(\hat{A}^*\hat{A} - A^*A\right)\mathcal{B}_T(x) \\
&= \sum_{j=1}^{\infty}\frac{1}{\lambda_T+\nu_j}\left\langle \phi_j, \left(\hat{A}^*\hat{A} - A^*A\right)\mathcal{B}_T\right\rangle_H \phi_j(x) \\
&= \sum_{j=1}^{\infty}\frac{1}{\lambda_T+\nu_j}\left\langle \phi_j, \left(\widehat{\tilde{A}}A - \tilde{A}A\right)\mathcal{B}_T\right\rangle_{L^2(\mathcal{X})} \phi_j(x), \qquad \text{(TR.6)}
\end{aligned}
$$

and $\left(\widehat{\tilde{A}}A - \tilde{A}A\right)\mathcal{B}_T(x)$

$$
\begin{aligned}
&= \int\left[\frac{1}{T}\sum_{t=1}^{T}\hat{f}(x|Z_t)I_t\Omega_t\hat{f}(\xi|Z_t) - \int f(x|z)I\left(z\in S^*\right)\Omega_0(z)f(\xi|z)f(z)dz\right]\mathcal{B}_T(\xi)d\xi \\
&=: \int I_T(x,\xi)\mathcal{B}_T(\xi)d\xi. \qquad \text{(TR.7)}
\end{aligned}
$$

From the uniform convergence of the kernel density estimator on S^*, and using the decomposition $\hat{f}(x,z) = \bar{f}(x,z) + \bar{b}(x,z) + f(x,z)$, where $\bar{f}(x,z) := \hat{f}(x,z) - E\left[\hat{f}(x,z)\right]$ and $\bar{b}(x,z) = E\left[\hat{f}(x,z)\right] - f(x,z)$, the dominant term in $I_T(x,\xi)$ is:

$$
\begin{aligned}
I_{T,1}(x,\xi) &= \int \frac{\hat{f}(x,z)I(z\in S^*)\Omega_0(z)\hat{f}(\xi,z)}{f(z)}dz - \int f(x|z)I\left(z\in S^*\right)\Omega_0(z)f(\xi|z)f(z)dz \\
&= \int \bar{f}(x,z)I(z\in S^*)\Omega_0(z)f\left(\xi|z\right)dz + \int \bar{b}(x,z)I(z\in S^*)\Omega_0(z)f\left(\xi|z\right)dz \\
&\quad + \int f(x|z)I(z\in S^*)\Omega_0(z)\bar{f}(\xi,z)dz + \int f(x|z)I(z\in S^*)\Omega_0(z)\bar{b}(\xi,z)dz \\
&\quad + \int \frac{\Delta\hat{f}(x,z)I(z\in S^*)\Omega_0(z)\Delta\hat{f}(\xi,z)}{f(z)}dz \\
&=: I_{T,11}(x,\xi) + I_{T,12}(x,\xi) + I_{T,13}(x,\xi) + I_{T,14}(x,\xi) + I_{T,15}(x,\xi).
\end{aligned}
$$

Using (TR.6) and (TR.7), we get the decomposition $\mathcal{E}_{T,2}(x) = \sum_{i=1}^{5}\mathcal{E}_{T,2i}(x)$. We focus on the contribution of $\mathcal{E}_{T,21}$ to $\mathcal{K}_T\left(\mathcal{E}_{T,2}(X), \mathcal{E}_{T,2}(X)\right)$ (the other terms can be bounded similarly). Using

$$
\begin{aligned}
&\int I_{T,11}(x,\xi)\mathcal{B}_T(\xi)d\xi \\
&= \frac{1}{T}\sum_{n=1}^{T}\int\left(K_{h_T}(x-X_n)K_{h_T}(z-Z_n) - E\left[K_{h_T}(x-X_n)K_{h_T}(z-Z_n)\right]\right)I\left(z\in S^*\right)\Omega_0(z)\left(A\mathcal{B}_T\right)(z)dz,
\end{aligned}
$$

the dominant term in $\mathcal{E}_{T,21}(X_t)$ is

$$\frac{1}{T}\sum_{n=1}^{T}\sum_{j=1}^{\infty}\frac{1}{\lambda_T+\nu_j}\left(\phi_j(X_n)I_n\Omega_n\left(A\mathcal{B}_T\right)(Z_n)-E\left[\phi_j(X)I\left(Z\in S^*\right)\Omega_0(Z)\left(A\mathcal{B}_T\right)(Z)\right]\right)\phi_j(X_t)$$
$$=:\frac{1}{T}\sum_{n=1}^{T}\eta_{n,t}.$$

Variable $\eta_{n,t}$ is such that

$$E\left[\eta_{n,t}|X_t,Z_t\right]=0 \qquad \text{(TR.8)}$$

for $n\neq t$. The contribution to $\mathcal{K}_T\left(\mathcal{E}_{T,2}(X),\mathcal{E}_{T,2}(X)\right)$ is

$$\frac{1}{T}\sum_t\frac{\Omega_t I_t}{\left(\sum_j K_{jt}\right)^2}\sum_{s\neq t}\sum_{u\neq s,t}K_{st}K_{ut}\left(\frac{1}{T}\sum_n\eta_{n,s}\right)\left(\frac{1}{T}\sum_m\eta_{m,u}\right).$$

The dominant term is

$$\frac{1}{T^3h_T^2}\sum_t\frac{\Omega_t I_t}{f(Z_t)^2}\sum_{s\neq t}\sum_{u\neq s,t}K_{st}K_{ut}\left(\frac{1}{T}\sum_n\eta_{n,s}\right)\left(\frac{1}{T}\sum_m\eta_{m,u}\right)$$
$$=\frac{1}{T^5h_T^2}\sum_n\sum_m\sum_s\sum_{u\neq s}a_{su}\eta_{n,s}\eta_{m,u}=:I$$

where

$$a_{su}=\sum_{t\neq s,u}\frac{\Omega_t I_t}{f(Z_t)^2}K_{st}K_{ut}.$$

To bound term I, let us compute

$$E[I^2]=\frac{1}{T^{10}h_T^4}\sum_{n_1}\sum_{m_1}\sum_{s_1}\sum_{u_1\neq s_1}\sum_{n_2}\sum_{m_2}\sum_{s_2}\sum_{u_2\neq s_2}E\left[a_{s_1u_1}a_{s_2u_2}\eta_{n_1,s_1}\eta_{m_1,u_1}\eta_{n_2,s_2}\eta_{m_2,u_2}\right].$$

Consider first the terms such that $n_1,m_1,n_2,m_2\neq s_1,u_1,s_2,u_2$. From (TR.8),

$$E\left[a_{s_1u_1}a_{s_2u_2}\eta_{n_1,s_1}\eta_{m_1,u_1}\eta_{n_2,s_2}\eta_{m_2,u_2}\right]=E\left[a_{s_1u_1}a_{s_2u_2}E\left[\eta_{n_1,s_1}\eta_{m_1,u_1}\eta_{n_2,s_2}\eta_{m_2,u_2}|X_{s_1},Z_{s_1},...,X_{u_2},Z_{u_2}\right]\right]$$

is different from zero only if the indices n_1,m_1,n_2,m_2 are either all equal, or such that there exist two pairs of equal indices. Let us for instance consider the term with $n_1=n_2=:n$, $m_1=m_2=:m$ and $n\neq m$:

$$E\left[a_{s_1u_1}a_{s_2u_2}E\left[\eta_{n,s_1}\eta_{n,s_2}\eta_{m,u_1}\eta_{m,u_2}|X_{s_1},Z_{s_1},...,X_{u_2},Z_{u_2}\right]\right]$$
$$=E\left[a_{s_1u_1}a_{s_2u_2}E\left[\eta_{n,s_1}\eta_{n,s_2}|X_{s_1},Z_{s_1},X_{s_2},Z_{s_2}\right]E\left[\eta_{m,u_1}\eta_{m,u_2}|X_{u_1},Z_{u_1},X_{u_2},Z_{u_2}\right]\right].$$

The contribution to $E[I^2]$ is

$$J=\frac{1}{T^{10}h_T^4}\sum_n\sum_m\sum_{s_1}\sum_{u_1\neq s_1}\sum_{s_2}\sum_{u_2\neq s_2}$$
$$E\left[a_{s_1u_1}a_{s_2u_2}E\left[\eta_{n,s_1}\eta_{n,s_2}|X_{s_1},Z_{s_1},X_{s_2},Z_{s_2}\right]E\left[\eta_{m,u_1}\eta_{m,u_2}|X_{u_1},Z_{u_1},X_{u_2},Z_{u_2}\right]\right].$$

Let us bound this term. Then,

$$E\left[\eta_{n,s_1}\eta_{n,s_2}|X_{s_1},Z_{s_1},X_{s_2},Z_{s_2}\right]=\sum_{j=1}^{\infty}\sum_{l=1}^{\infty}\frac{1}{\lambda_T+\nu_j}\frac{1}{\lambda_T+\nu_l}c_{jl}\phi_j(X_{s_1})\phi_l(X_{s_2}),$$

where

$$c_{jl}=E\left[\phi_j(X_n)\phi_l(X_n)I_n\Omega_n^2\left(A\mathcal{B}_T\right)(Z_n)^2\right]-E\left[\phi_j(X_n)I_n\Omega_n\left(A\mathcal{B}_T\right)(Z_n)\right]E\left[\phi_l(X_n)I_n\Omega_n\left(A\mathcal{B}_T\right)(Z_n)\right].$$

We get:

$$\begin{aligned}
&E\left[a_{s_1u_1}a_{s_2u_2}E\left[\eta_{n,s_1}\eta_{n,s_2}|X_{s_1},Z_{s_1},X_{s_2},Z_{s_2}\right]E\left[\eta_{m,u_1}\eta_{m,u_2}|X_{u_1},Z_{u_1},X_{u_2},Z_{u_2}\right]\right]\\
=\;&\sum_{j=1}^{\infty}\sum_{l=1}^{\infty}\sum_{k=1}^{\infty}\sum_{p=1}^{\infty}\frac{1}{\lambda_T+\nu_j}\frac{1}{\lambda_T+\nu_l}c_{jl}\frac{1}{\lambda_T+\nu_k}\frac{1}{\lambda_T+\nu_p}c_{kp}\\
&\sum_{t_1\neq s_1,u_1}\sum_{t_2\neq s_2,u_2,t_1}E\left[\frac{\Omega_{t_1}I_{t_1}}{f(Z_{t_1})^2}\frac{\Omega_{t_2}I_{t_2}}{f(Z_{t_2})^2}K_{s_1t_1}K_{u_1t_1}K_{s_2t_2}K_{u_2t_2}\phi_j(X_{s_1})\phi_l(X_{s_2})\phi_k(X_{u_1})\phi_p(X_{u_2})\right].
\end{aligned}$$

Now, for a term with $s_1\neq s_2\neq u_1\neq u_2$ we have:

$$\begin{aligned}
&E\left[\frac{\Omega_{t_1}I_{t_1}}{f(Z_{t_1})^2}\frac{\Omega_{t_2}I_{t_2}}{f(Z_{t_2})^2}K_{s_1t_1}K_{u_1t_1}K_{s_2t_2}K_{u_2t_2}\phi_j(X_{s_1})\phi_l(X_{s_2})\phi_k(X_{u_1})\phi_p(X_{u_2})\right]\\
=\;&E\left[\frac{\Omega_{t_1}I_{t_1}}{f(Z_{t_1})^2}\frac{\Omega_{t_2}I_{t_2}}{f(Z_{t_2})^2}K_{s_1t_1}K_{u_1t_1}K_{s_2t_2}K_{u_2t_2}\left(A\phi_j\right)(Z_{s_1})\left(A\phi_l\right)(Z_{s_2})\left(A\phi_k\right)(Z_{u_1})\left(A\phi_p\right)(Z_{u_2})\right]\\
=\;&O\left(h_T^4E\left[\Omega_{t_1}I_{t_1}\Omega_{t_2}I_{t_2}A\phi_j(Z_{t_1})A\phi_k(Z_{t_1})A\phi_l(Z_{t_2})A\phi_p(Z_{t_2})\right]\right),
\end{aligned}$$

and

$$\begin{aligned}
&E\left[\Omega_{t_1}I_{t_1}\Omega_{t_2}I_{t_2}A\phi_j(Z_{t_1})A\phi_k(Z_{t_1})A\phi_l(Z_{t_2})A\phi_p(Z_{t_2})\right]\\
=\;&E\left[\Omega_{t_1}I_{t_1}A\phi_j(Z_{t_1})A\phi_k(Z_{t_1})\right]E\left[\Omega_{t_2}I_{t_2}A\phi_l(Z_{t_2})A\phi_p(Z_{t_2})\right]=\nu_j\nu_l\delta_{jk}\delta_{lp}.
\end{aligned}$$

Therefore we get:

$$J=O\left(T^2h_T^4\sum_{j=1}^{\infty}\sum_{l=1}^{\infty}\frac{\nu_j}{\left(\lambda_T+\nu_j\right)^2}\frac{\nu_l}{\left(\lambda_T+\nu_l\right)^2}c_{jl}^2\right).$$

By similar arguments for the other contributions to $E[I^2]$, we get:

$$E[I^2]=O\left(\frac{1}{T^2}\sum_{j=1}^{\infty}\sum_{l=1}^{\infty}\frac{\nu_j}{\left(\lambda_T+\nu_j\right)^2}\frac{\nu_l}{\left(\lambda_T+\nu_l\right)^2}c_{jl}^2\right).$$

To bound the term in the RHS, we use that:

$$\begin{aligned}
|c_{jl}|\;\leq\;&\sup_{j,l\in\mathbb{N}}\sup_{z\in S^*}E\left[\left|\phi_j(X)\phi_l(X)\right||Z=z\right]\sup_{z\in S^*}\Omega(z)E\left[I_n\Omega_n\left(A\mathcal{B}_T\right)(Z_n)^2\right]\\
&+\left(\sup_{j\in\mathbb{N}}\sup_{z\in S^*}E\left[\left|\phi_j(X)\right||Z=z\right]\sup_{z\in S^*}\Omega(z)^{1/2}E\left[I_n\Omega_n^{1/2}\left|\left(A\mathcal{B}_T\right)(Z_n)\right|\right]\right)^2\\
\leq\;&2\sup_{j\in\mathbb{N}}\sup_{z\in S^*}E\left[\phi_j(X)^2|Z=z\right]\sup_{z\in S^*}\Omega(z)Q_{\lambda_T}=O\left(\lambda_T^{1+\beta}\right)=O\left(\lambda_T\right),
\end{aligned}$$

from Assumption A.7 (iii) and Appendix A.2.3. Thus we get:

$$
\begin{aligned}
E[I^2] &= O\left(\frac{\lambda_T^2}{T^2}\left(\sum_{j=1}^{\infty}\frac{\nu_j}{(\lambda_T+\nu_j)^2}\right)^2\right) = O\left(\frac{1}{T^2}\left(\sum_{j=1}^{\infty}\frac{\nu_j}{(\lambda_T+\nu_j)}\right)^2\right) \\
&= O\left(\frac{1}{T^2}\log\left(1/\lambda_T\right)^2\right),
\end{aligned}
$$

using an argument as in Section B.5.1. We deduce

$$
I = O_p\left(\frac{1}{T}\log\left(1/\lambda_T\right)\right) = o_p\left(\frac{1}{Th_T^{1/2}}\right).
$$

The conclusion follows.

6 Proof of Lemma B.6

We provide a detailed proof for the bound of $\mathcal{K}_T\left(U - \mathcal{B}_T(X), \mathcal{E}_{T,1}(X)\right)$. Using the notation in the proof of Lemma B.3, we have

$$
\mathcal{K}_T\left(U - \mathcal{B}_T(X), \mathcal{E}_{T,1}(X)\right) = -\mathcal{K}_T\left(b, \mathcal{E}_{T,1}(X)\right) + \mathcal{K}_T\left(U - \eta, \mathcal{E}_{T,1}(X)\right) =: -J_{41,T} + J_{42,T}.
$$

Let us first consider $J_{41,T}$. Similar arguments as in the proof of Lemma B.5, Section B.5.1, show that

$$
\begin{aligned}
J_{41,T} &= \frac{1}{T^2}\sum_t H_0(Z_t)^{-1} I_t \sum_{n\neq t} U_n \left(\frac{1}{T^2h_T^2}\sum_{s\neq t}\sum_{u\neq t,s} K_{st}K_{ut}b_sQ_{un}\right) \\
&\quad + \frac{1}{T^2}\sum_t H_0(Z_t)^{-1} I_t \sum_{n\neq t} G_{n,T} \left(\frac{1}{T^2h_T^2}\sum_{s\neq t}\sum_{u\neq t,s} K_{st}K_{ut}b_sQ_{un}\right) + o_p((Th_T^{1/2})^{-1}) \\
=: &\quad \frac{1}{T^2}J_{41,T}^{*} + \frac{1}{T^2}J_{41,T}^{**} + o_p((Th_T^{1/2})^{-1}).
\end{aligned}
$$

Furthermore, $J_{41,T}^{*} = \sum_n a_n U_n$ where

$$
a_n = \sum_{t\neq n} H_0(Z_t)^{-1} I_t \left(\frac{1}{T^2h_T^2}\sum_{s\neq t}\sum_{u\neq t,s} K_{st}K_{ut}b_sQ_{un}\right).
$$

We have $E\left[\left(J_{41,T}^{*}\right)^2\right] = \sum_n E\left[a_n^2U_n^2\right] = \sum_n E\left[a_n^2 V_0\left(Z_n\right)\right]$. To simplify, let $\Omega_0(z) = V_0(z)^{-1} = 1$. Using an argument similar as for the derivation of (TR.3), $E\left[a_n^2\right]$ is asymptotically equivalent to $\sum_{t\neq n}\sum_{i\neq t,n} E\left[I_tI_ib_tb_iE\left[Q_{nt}Q_{ni}|Z_t,Z_i\right]\right]$. Using (TR.4), (TR.5) and

Cauchy-Schwarz inequality, for $t \neq i$ we get

$$\begin{aligned}
& E\left[I_t I_i b_t b_i E\left[Q_{nt} Q_{ni} | Z_t, Z_i\right]\right] \\
\leq & \left\{\sum_{j=1}^{\infty} \frac{\nu_j^2}{\left(\lambda_T+\nu_j\right)^2}+O(h_T^2) \sum_{j=1}^{\infty} \frac{\nu_j}{\lambda_T+\nu_j} \sum_{l=1}^{\infty} \frac{\sqrt{\nu_l}}{\lambda_T+\nu_l}+O(h_T^4)\left(\sum_{j=1}^{\infty} \frac{\sqrt{\nu_j}}{\lambda_T+\nu_j}\right)^2\right\} E\left[I_t b_t^2\right] \\
=: & S_1(\lambda_T) E\left[I_t b_t^2\right].
\end{aligned}$$

Thus, $E\left[\left(J_{41,T}^*\right)^2\right]=O\left(T^3 S_1(\lambda_T) E\left[I_t b_t^2\right]\right)$ and $\frac{1}{T^2} J_{41,T}^*=O_p\left(\frac{\sqrt{h_T^{1/2} S_1(\lambda_T)}}{\sqrt{T h_T^{1/2}}} E\left[I_t b_t^2\right]^{1/2}\right)$. Similarly, writing $J_{41,T}^{**}=\sum_n a_n G_{n,T}$ and using $a_n=O_p\left(T \sqrt{S_1(\lambda_T) E\left[I_t b_t^2\right]}\right)$, $G_{n,T}=O_p(h_T^2)$, uniformly in n, we get $\frac{1}{T^2} J_{41,T}^{**}=O_p\left(h_T^2 \sqrt{S_1(\lambda_T)} E\left[I_t b_t^2\right]^{1/2}\right)$. Now, using that $\sum_{l=1}^{\infty} \frac{\sqrt{\nu_l}}{\lambda_T+\nu_l} \leq \left(\sum_{l=1}^{\infty} \frac{\nu_l l^2}{\left(\lambda_T+\nu_l\right)^2}\right)^{1/2}\left(\sum_{l=1}^{\infty} \frac{1}{l^2}\right)^{1/2}=O\left(\frac{1}{\lambda_T^{1/2}} \log(1/\lambda_T)\right)$ under Assumption B.7 (i) (see Lemma A.6 is GS) we get $S_1(\lambda_T)=O\left(\log(1/\lambda_T)^2\right)$ from Assumption 4. Thus, $h_T^2 \sqrt{S_1(\lambda_T)}=o\left(\frac{1}{\sqrt{T h_T^{1/2}}}\right)$ from Assumption 3, and $J_{41,T}=o_p\left(\frac{1}{\sqrt{T h_T^{1/2}}} Q_{\lambda_T}^{1/2}\right)+o_p((T h_T^{1/2})^{-1})$.

Let us now consider $J_{42,T}$. By similar arguments as above we have

$$\begin{aligned}
J_{42,T} &= \frac{1}{T^3 h_T} \sum_t H_0(Z_t)^{-1} I_t \sum_{s \neq t} \sum_{n \neq s,t}\left(U_s-\eta_s\right) U_n\left(\frac{1}{T h_T} \sum_{u \neq t,s} K_{st} K_{ut} Q_{un}\right)+o_p((T h_T^{1/2})^{-1}) \\
&=: \frac{1}{T^3 h_T} J_{42,T}^*+o_p((T h_T^{1/2})^{-1}),
\end{aligned}$$

where $J_{42,T}^*=\sum_s \sum_{n \neq s} d_{ns}\left(U_s-\eta_s\right) U_n$ and $d_{ns}:=\sum_{t \neq s,n} H_0(Z_t)^{-1} I_t\left(\frac{1}{T h_T} \sum_{u \neq t,s} K_{st} K_{ut} Q_{un}\right)$. Using that $E\left[U_s | \mathcal{I}, W_u\right]=E\left[\eta_s | \mathcal{I}, W_u\right]=0$ for $s \neq u$, we get

$$E\left[\left(J_{42,T}^*\right)^2\right]=\sum_s \sum_{n \neq s} E\left[d_{ns}^2 \Psi_1\left(Z_s\right)\right]+\sum_s \sum_{n \neq s} E\left[d_{ns} d_{sn} \Psi_2\left(Z_s\right) \Psi_2\left(Z_n\right)\right],$$

where $\Psi_1\left(Z_s\right):=E\left[\left(U_s-\eta_s\right)^2 | Z_s\right]$, $\Psi_2\left(Z_s\right):=E\left[\left(U_s-\eta_s\right) U_s | Z_s\right]$. Then, $E\left[d_{ns}^2 \Psi_1\left(Z_s\right)\right]$ is asymptotically equivalent to $\sum_{t \neq s,n} \sum_{i \neq t,s,n} E\left[I_t I_i \Psi_1\left(Z_s\right) K_{st} K_{si} Q_{nt} Q_{ni}\right]$. Using (TR.4), (TR.5), $E\left[\Psi_1\left(Z_s\right) K_{st} K_{si} | Z_i, Z_t\right]=O_p\left(h_T K * K\left(\frac{Z_i-Z_t}{h_T}\right) f(Z_t) \Psi_1(Z_t)\right)$ and Cauchy-Schwarz inequality, we get $E\left[d_{ns}^2 \Psi_1\left(Z_s\right)\right]=O\left(T^2 h_T^2 S_1(\lambda_T)\right)$. A similar bound holds for $E\left[d_{ns} d_{sn} \Psi_2\left(Z_s\right) \Psi_2\left(Z_n\right)\right]$. Then, $J_{42,T}=o_p((T h_T^{1/2})^{-1})$ using the same arguments as for $J_{41,T}$.

7 Proof of Lemma B.7

We have:

$$\begin{aligned}\mathcal{K}_T\left(\mathcal{R}_T(X),\mathcal{R}_T(X)\right) &= \frac{1}{T}\sum_t \frac{\Omega_t I_t}{\left(\sum_j K_{jt}\right)^2}\sum_{s\neq t}\sum_{u\neq t} K_{st}K_{ut}\mathcal{R}_T(X_s)\mathcal{R}_T(X_u)\\ &\quad -\frac{1}{T}\sum_t \frac{\Omega_t I_t}{\left(\sum_j K_{jt}\right)^2}\sum_{s\neq t} K_{st}^2\mathcal{R}_T(X_s)^2\\ &=: I_{1,T}-I_{2,T}.\end{aligned}$$

Let us first consider $I_{1,T}$. We have:

$$I_{1,T}=\frac{1}{T}\sum_t \frac{\Omega_t I_t}{\left(\sum_j K_{jt}\right)^2}\left(\sum_{s\neq t}K_{st}\mathcal{R}_T(X_s)\right)^2 \leq \max_{t\in\mathcal{T}^*}\left|\frac{(Th_T)^2\,\Omega_t}{\left(\sum_j K_{jt}\right)^2}\right|\sup_{z\in S^*}\left(\frac{1}{Th_T}\sum_s K\left(\frac{Z_s-z}{h_T}\right)\mathcal{R}_T(X_s)\right)^2.$$

Since $\max_{t\in\mathcal{T}^*}\left|\frac{(Th_T)^2\,\Omega_t}{\left(\sum_j K_{jt}\right)^2}\right| = O_p(1)$, we get $I_{1,T}=o_p\left(1/\left(Th_T^{1/2}\right)\right)$ from Assumption A.6 (i).

Let us now consider $I_{2,T}$. We have:

$$I_{2,T}\leq \max_{t\in\mathcal{T}^*}\left|\frac{(Th_T)^2\,\Omega_t I}{\left(\sum_j K_{jt}\right)^2}\right| K(0)\frac{1}{Th_T}\sup_{z\in S^*}\left(\frac{1}{Th_T}\sum_{s\neq t}K\left(\frac{Z_s-z}{h_T}\right)\mathcal{R}_T(X_s)^2\right).$$

We get $I_{2,T}=o_p\left(1/\left(Th_T^{1/2}\right)\right)$ from Assumptions A.6 (ii) and 3. The conclusion follows.

8 Proof of Lemma B.8

We provide detailed proofs for the bounds of $\mathcal{K}_T\left(\mathcal{R}_T(X),U-\mathcal{B}_T(X)\right)$ and $\mathcal{K}_T\left(\mathcal{R}_T(X),\mathcal{E}_{T,1}(X)\right)$.

8.1 Bound of $\mathcal{K}_T\left(\mathcal{R}_T(X),U-\mathcal{B}_T(X)\right)$

Write $\mathcal{K}_T\left(\mathcal{R}_T(X),U-\mathcal{B}_T(X)\right) = \frac{1}{T}\sum_t \frac{\Omega_t I_t}{\left(\sum_j K_{jt}\right)^2}\sum_{s\neq t}K_{st}\mathcal{R}_T(X_s)\Phi_{t,s}$, where we set $\Phi_{t,s} := \sum_{u\neq t,s}K_{ut}\left(U_u-\mathcal{B}_T(X_u)\right)$. By applying twice the Cauchy-Schwarz inequality, we get

$$\begin{aligned}&\left|\mathcal{K}_T\left(\mathcal{R}_T(X),U-\mathcal{B}_T(X)\right)\right|\\ \leq\;& \max_{t\in\mathcal{T}^*}\left|\frac{(Th_T)^2\Omega_t}{(\sum_j K_{jt})^2}\right|\frac{1}{T^3h_T^2}\left(\sum_t\sum_{s\neq t}K_{st}\mathcal{R}_T(X_s)^2 I_t\right)^{1/2}\left(\sum_t\sum_{s\neq t}K_{st}\Phi_{t,s}^2 I_t\right)^{1/2}\\ \leq\;& \max_{t\in\mathcal{T}^*}\left|\frac{(Th_T)^2\Omega_t}{(\sum_j K_{jt})^2}\right|\frac{1}{Th_T^{3/2}}\left(\sup_{z\in S^*}\frac{1}{Th_T}\sum_{s\neq t}K\left(\frac{Z_s-z}{h_T}\right)\mathcal{R}_T(X_s)^2 I_t\right)^{1/2}\left(\frac{1}{T^2}\sum_t\sum_{s\neq t}K_{st}\Phi_{t,s}^2 I_t\right)^{1/2}\end{aligned}$$

Using $\max_{t\in\mathcal{T}^*}\left|\frac{(Th_T)^2\Omega_t}{(\sum_j K_{jt})^2}\right| = O_p(1)$ and Assumption A.6 (ii), $\mathcal{K}_T(\mathcal{R}_T(X), U-\mathcal{B}_T(X)) = o_p\left(1/(Th_T^{1/2})\right)$ follows if we can show that $E\left[K_{st}\Phi_{t,s}^2 I_t\right] = O\left(Th_T^2\right)$, uniformly in $s\neq t$. By using the notation in the proof of Lemma B.3 we have

$$\Phi_{t,s} = \sum_{u\neq t,s} K_{ut}\left(U_u-\eta_u\right) - \sum_{u\neq t,s} K_{ut} b_u =: \Phi_{1,ts} - \Phi_{2,ts}.$$

Since variables $U_u - \eta_u$ are uncorrelated conditionally on $\mathcal{I}$, $E\left[K_{st}\Phi_{1,ts}^2 I_t\right] = \sum_{u\neq t,s} E\left[I_t K_{st} K_{ut}^2 \left(U_u-\eta_u\right)^2\right] = O\left(Th_T^2\right)$, uniformly in $s\neq t$. Furthermore, $E\left[K_{st}\Phi_{2,ts}^2 I_t\right] = O\left(T^2 h_T^3 E\left[I_t b_t^2\right]\right) = O\left(T^2 h_T^3 Q_{\lambda_T}\right) = o\left(Th_T^2\right)$ by Assumptions 3 and 4 (see Appendix A.2.3).

8.2 Bound of $\mathcal{K}_T\left(\mathcal{R}_T(X), \mathcal{E}_{T,1}(X)\right)$

By the same argument as in Section B.8.1, $\mathcal{K}_T\left(\mathcal{R}_T(X), \mathcal{E}_{T,1}(X)\right) = o_p\left(1/(Th_T^{1/2})\right)$ follows if we can show that $E\left[K_{st}\Phi_{3,ts}^2 I_t\right] = O\left(Th_T^2\right)$, uniformly in $s\neq t$, where $\Phi_{3,t,s} := \sum_{u\neq t,s} K_{ut}\mathcal{E}_{T,1}(X_u)$. As in the proof of Lemma B.5 (see Section B.5.1) we have

$$\begin{aligned}\Phi_{3,ts} &= h_T\sum_n U_n\left(\frac{1}{Th_T}\sum_{u\neq t,s} K_{ut}Q_{un}\right) + \frac{1}{T}\sum_n\sum_{u\neq t,s} K_{ut}U_n V_{un} \\ &\quad + h_T\sum_n G_{n,T}\left(\frac{1}{Th_T}\sum_{u\neq t,s} K_{ut}\Psi_{un}\right) \quad =: \quad \Phi_{31,ts} + \Phi_{32,ts} + \Phi_{33,ts}.\end{aligned}$$

From (TR.4) and (TR.5), $E\left[\left(\frac{1}{Th_T}\sum_{u\neq t,s} K_{ut}Q_{un}\right)^2\right]$ and $E\left[\left(\frac{1}{Th_T}\sum_{u\neq t,s} K_{ut}\Psi_{un}\right)^2\right]$ are asymptotically equivalent to

$$E\left[Q_{tn}^2\right] = \sum_{j=1}^{\infty}\frac{\nu_j^2}{\left(\lambda_T+\nu_j\right)^2} + O(h_T^2)\sum_{j=1}^{\infty}\frac{\nu_j^{3/2}}{\left(\lambda_T+\nu_j\right)^2} + O(h_T^4)\sum_{j=1}^{\infty}\frac{\nu_j}{\left(\lambda_T+\nu_j\right)^2} =: S_2\left(\lambda_T\right).$$

Using Cauchy-Schwarz inequality, Assumptions A.7 and 4, and similar arguments as in the proof of Lemma B.5 we get $S_2\left(\lambda_T\right) = O\left(\log\left(1/\lambda_T\right)\right)$. Thus, $E\left[\Phi_{31,ts}^2 K_{st} I_t\right] = O\left(Th_T^2\log\left(1/\lambda_T\right)\right)$ and $E\left[\Phi_{33,ts}^2 K_{st} I_t\right] = O_p\left(Th_T^3\sqrt{\log\left(1/\lambda_T\right)}\right)$. Moreover, $E\left[\Phi_{32,ts}^2 K_{st} I_t\right] = O\left(h_T^2/\lambda_T\right)$. From Assumptions 3 and 4, the conclusion follows.

9 Proof of Lemma C.1

The proof is similar to the one of Lemma B.1, by using the split (15) and $\frac{1}{T}\sum_t \upsilon_t^2 = O_p(E[\upsilon^2])$, $E[\upsilon^2]^{1/2} \leq E\left[\left|Y-\varphi_{\lambda_T}(X)\right|^m\right]^{1/m} + E\left[\left|Y-r(Z)\right|^m\right]^{1/m} = O(1)$ from Assumption A.2 (ii).

10 Proof of Lemma C.2

By a similar argument as in the proof of Lemma B.2 and using the split (15), the dominant contribution in $\xi_{3,T}$ is given by

$$\xi^*_{32,T} = \frac{1}{T}\sum_t \sum_{s\neq t} \frac{\Omega_t K(0)}{(\sum_j K_{jt})^2} \upsilon_t \upsilon_s K_{st} I_t.$$

Define $\bar{\eta}_s := \upsilon_s - \bar{b}_s$ and $\bar{b}_s := E[\upsilon_s|Z_s] = b_s$. Then:

$$\begin{aligned}
\xi^*_{32,T} &= \frac{1}{T}\sum_t \sum_{s\neq t} \frac{\Omega_t K(0)}{(\sum_j K_{jt})^2} \bar{b}_t \bar{b}_s K_{st} I_t + \frac{1}{T}\sum_t \sum_{s\neq t} \frac{\Omega_t K(0)}{(\sum_j K_{jt})^2} \bar{\eta}_t \bar{\eta}_s K_{st} I_t \\
&\quad + \frac{1}{T}\sum_t \sum_{s\neq t} \frac{\Omega_t K(0)}{(\sum_j K_{jt})^2} \bar{b}_t \bar{\eta}_s K_{st} I_t + \frac{1}{T}\sum_t \sum_{s\neq t} \frac{\Omega_t K(0)}{(\sum_j K_{jt})^2} \bar{\eta}_t \bar{b}_s K_{st} I_t \\
&=: \xi^*_{321,T} + \xi^*_{322,T} + \xi^*_{323,T} + \xi^*_{324,T}.
\end{aligned}$$

From the proof of Lemma B.2, $\xi^*_{321,T} = \xi_{321,T} = O_p\left(\frac{1}{Th_T} Q_{\lambda_T}\right)$. Similarly, $\xi^*_{322,T} = O_p\left(\frac{1}{T^2 h_T^{3/2}} E\left[\bar{\eta}_t^2\right]\right) = O_p\left(\frac{1}{T^2 h_T^{3/2}} E[\varphi_{\lambda_T}(X_t)^2]\right)$. Using $E[\varphi_{\lambda_T}(X)^2]^{1/2} \leq E[\varphi_0(X)^2]^{1/2} + E\left[|Y - \varphi_0(X)|^m\right]^{1/m} + E\left[\left|Y - \varphi_{\lambda_T}(X)\right|^m\right]^{1/m} = O(1)$ from Assumptions A.1 and A.2 (ii), we get $\xi^*_{322,T} = O_p\left(\frac{1}{T^2 h_T^{3/2}}\right)$. The other terms are bounded similarly, and the conclusion follows.

11 Proof of Lemma C.3

By using the split (15) and the definitions $\bar{\eta}_s := \upsilon_s - \bar{b}_s$ and $\bar{b}_s := E[\upsilon_s|Z_s] = b_s$, we have:

$$\mathcal{K}_T(\upsilon, \upsilon) = \mathcal{K}_T\left(\bar{b}, \bar{b}\right) + 2\mathcal{K}_T\left(\bar{b}, \bar{\eta}\right) + \mathcal{K}_T(\bar{\eta}, \bar{\eta}) =: J^*_{11,T} + J^*_{12,T} + J^*_{13,T}.$$

From the proof of Lemma B.3, $J^*_{11,T} = J_{11,T} = Q_{\lambda_T}(1 + o_p(1))$ and $J^*_{13,T} = O_p\left(\frac{1}{Th_T^{1/2}} E\left[\bar{\eta}_t^2\right]\right) = O_p\left(\frac{1}{Th_T^{1/2}} E[\varphi_{\lambda_T}(X_t)^2]\right) = O_p\left(\frac{1}{Th_T^{1/2}}\right)$. Term $J^*_{12,T}$ is bounded similarly, and the conclusion follows.

12 Proof of Lemma C.4

Using the same notation as in the proof of Lemma C.3, we have:

$$\mathcal{K}_T(U^*, \upsilon) = \mathcal{K}_T\left(U^*, \bar{b}\right) + \mathcal{K}_T(U^*, \bar{\eta}) =: J^*_{21,T} + J^*_{22,T}.$$

By the same argument as for term $J_{21,T}$ in the proof of Lemma B.4, we have $J^*_{21,T} = O_p\left(\frac{1}{\sqrt{T}} Q_{\lambda_T}\right)$. The term $J^*_{22,T}$ can be bounded by a similar argument as term $J^*_{13,T}$ in the proof of Lemma C.3. We get $J^*_{13,T} = O_p\left(\frac{1}{Th_T^{1/2}}E[\varphi_{\lambda_T}(X_t)^2]^{1/2}\right) = O_p\left(\frac{1}{Th_T^{1/2}}\right)$. The conclusion follows.

13 Proof of Lemma D.1

From Cauchy-Schwarz inequality,

$$\left|\hat{V}(Z_t) - V_0(Z_t)\right| \leq \left|\sum_j w_{tj}U_j^2 - V_0(Z_t)\right| + 2A\left(Z_t\right) + B\left(Z_t\right),$$

where $A\left(Z_t\right) = \left(\sum_j w_{tj}U_j^2\right)^{1/2}\left(\sum_j w_{tj}\left|\Delta\bar{\varphi}(X_j)\right|^2\right)^{1/2}$, $B\left(Z_t\right) = \sum_j w_{tj}\left|\Delta\bar{\varphi}(X_j)\right|^2$, and $\Delta\bar{\varphi} = \bar{\varphi} - \varphi_0$. As in the proof of Lemma C.2 in TK and using $h_T = \bar{c}T^{-\bar{\eta}}$ with $\bar{\eta} < 1 - 4/m$, $\sup_{Z_t \in S_*}\left|\sum_j w_{tj}U_j^2 - V_0(Z_t)\right| = O_p\left(\sqrt{\frac{\log T}{Th_T}} + h_T^2\right)$. Further, from Lemma C.6 of TK and Assumption A.2 (i), $\sup_{Z_t \in S_*}\sum_j w_{tj}U_j^2 = o_p\left(T^{2/m}\right)$. Then, (i) follows from Assumption A.10 and uniform convergence of $\hat{f}(z)$ over S_*. Points (ii) and (iii) follow from (i), Assumption A.9, and uniform convergence of $\hat{f}(z)$ over S_*.

14 Proof of Lemma D.2

The structure of the proof is the same as for the proof of Proposition 2 in Appendix 3. We highlight the major changes. Let us first consider the asymptotic behavior of $\bar{\xi}^*_{5,T} = \frac{1}{T}\sum_t \frac{\hat{\Omega}_t I_t}{\left(\sum_j K_{jt}\right)^2}\sum_{s\neq t}\sum_{u\neq t,s}K_{st}K_{ut}U^*_s U^*_u$. We have $\bar{\xi}_{5,T} = \frac{1}{T^3h_T^2}\sum_t H_{\lambda_T}(Z_t)^{-1}I_t$ $\sum_{s\neq t}\sum_{u\neq t,s}K_{st}K_{ut}U^*_s U^*_u + O_p\left(\frac{\log T}{Th_T}\sup_{z\in S_*}\left|\hat{H}(z)^{-1} - H_{\lambda_T}(z)^{-1}\right|\right)$, where $H_{\lambda_T}(z) = V_{\lambda_T}(z)f(z)^2$. The first term is $O_p\left(\left(Th_T^{1/2}\right)^{-1}\right)$ from Assumption A.11. Using the uniform convergence of the kernel estimator $\hat{f}$, Assumptions A.11 and A.13, and $\inf_{S^*}\Omega_0 > 0$, we get $\sup_{z\in S_*}\left|\hat{H}(z)^{-1} - H_{\lambda_T}(z)^{-1}\right| = O_p\left(\sqrt{\frac{\log T}{Th_T}} + h_T^2\right) + o_p\left(T^{-1/6}\right)$. Then, from $h_T = \bar{c}T^{-\bar{\eta}}$ with $2/9 < \bar{\eta} < \min\{1 - 4/m, 1/3\}$, we get $Th_T^{1/2}\bar{\xi}^*_{5,T} = O_p(1)$. Let us now consider the proof of the technical Lemmas C.1-C.4. These proofs are virtually unchanged, and rely on the uniform convergence of $\hat{\Omega}(z)$ to $\Omega_{\lambda_T}(z)$ (Assumptions A.11 and A.13), and on the uniform bound $\Omega_{\lambda_T}(z) \leq c_2\Omega_0(z)$ (Assumption A.13).

15 Proof of Lemma D.3

Since $\ker(A^*)^{\perp} = \overline{\text{Range}(A)}$, and the norms $L^2(\mathcal{Z})$ and $L^2_{\lambda_T}(\mathcal{Z})$ are equivalent under Assumption A.11, the conclusion follows.