

Empirical Asset Pricing

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Conditional Asset Pricing: Tests and Critiques

Lecture Outline

1. Jagannathan and Wang: cond. CAPM
2. Lettau and Ludvigson: cond. Consumption CAPM
3. Lewellen and Nagel's critique
4. Characteristics are Covariances

Relevant readings:

- Jagannathan and Wang, 1996, "The conditional CAPM and the Cross-Section of Expected Stock Returns", Journal of Finance
- Lettau and Ludvigson, 2001, "Resurrecting the (C)CAPM: A Cross-Sectional Test When Risk Premia Are Time-Varying", Journal of Political Economy
- Cochrane, chapters 8, 9, and 13

- Lewellen and Nagel, 2006, “The conditional CAPM does not explain asset-pricing anomalies”, Journal of Financial Economics
- Kelly, Pruitt, and Su, 2019, “Characteristics are covariances: A unified model of risk and return”, Journal of Financial Economics
- Also: take a look at the papers cited in these class notes

1. Jagannathan and Wang, 1996

Pitfalls of Unconditional Tests

- CAPM is severely rejected by Fama and French's results
- However, the unconditional tests of CAPM do not take two elements into account:
 1. Betas vary over-time in a way that is correlated with expected risk premia. This fact prevents an unconditional version of CAPM from holding. For example: distressed firms may be more risky in bad times
 2. The aggregate wealth portfolio, which the CAPM postulates to be the market portfolio, may be very different from the equity market proxy used in most studies
- The goal of the paper is to account for these issues and see whether a conditional version of CAPM holds
- The paper is interesting because it is one of the first attempts at specifying a conditional CAPM and because of its implementation of GMM tests of AP models

The Conditional CAPM

- For CAPM to hold, one needs to assume that hedging motives for future consumption (Merton, 1973) are not strong
- It is a two-period model
- But it holds period by period: based on the information available to investors at a given point in time

$$E(R_{it}|I_{t-1}) = \gamma_{0t-1} + \gamma_{1t-1}\beta_{it-1} \quad (1)$$

where γ_{0t-1} is the conditional zero-beta rate, γ_{1t-1} is the conditional market risk premium, and β is computed using conditional moments

$$\beta_{it-1} = \frac{Cov(R_{it}, R_{mt}|I_{t-1})}{Var(R_{mt}|I_{t-1})}$$

- Average asset returns (over a time-series) proxy for the unconditional expected return on an asset
- Hence, one needs to derive an expression for unconditional returns

- Take unconditional expectations of each side of (1):

$$E(R_{it}) = \gamma_0 + \gamma_1 \bar{\beta}_i + Cov(\gamma_{1t-1}, \beta_{it-1}) \quad (2)$$

where $\gamma_0 = E(\gamma_{0t-1})$, $\gamma_1 = E(\gamma_{1t-1})$, and $\bar{\beta}_i = E(\beta_{it-1})$

- Main issue: $Cov(\gamma_{1t-1}, \beta_{it-1}) \neq 0$
- Hence, you cannot obtain an unconditional CAPM

The beta-premium sensitivity

- One can do a linear projection of the conditional beta β_{it-1} onto γ_{1t-1}
- The coefficient in this linear projection is $\theta_i = \frac{Cov(\beta_{it-1}, \gamma_{1t-1})}{Var(\gamma_{1t-1})}$, which is called the beta-premium sensitivity

- Then, one can express (2) as

$$E(R_{it}) = \gamma_0 + \gamma_1 \bar{\beta}_i + \theta_i Var(\gamma_{1t-1}) \quad (3)$$

- The assets with higher expected returns are those with:
 - Higher average beta (\approx unconditional beta)
 - Higher sensitivity of their beta to changes in the market risk premium
- Notice: the parameter that links θ_i to expected returns is not free, it is $Var(\gamma_{1t-1})$ (see Lewellen and Nagel, 2006)

A two-beta representation

- Because conditional betas are not observable, one cannot easily estimate θ_i
- Then, need to express the model in terms of sensitivity of **returns** to changes in market risk premium
- The authors show that, under fairly mild assumptions, (3) can be expressed as a two-beta model

$$E(R_{it}) = a_0 + a_1\beta_i + a_2\beta_i^\gamma \quad (4)$$

where

$$\begin{aligned}\beta_i &= \text{Cov}(R_{it}, R_{mt}) / \text{Var}(R_{mt}) \\ \beta_i^\gamma &= \text{Cov}(R_{it}, \gamma_{1t-1}) / \text{Var}(\gamma_{1t-1})\end{aligned}$$

the first beta is the unconditional beta, the second one is the premium-beta, which measures beta-instability risk

- This model is neither an ICAPM nor an APT. It is derived from conditional CAPM

Empirical Specification

- The model still needs to be made closer to the data
- In particular, one needs to provide operational specifications of the two betas
- To the purpose of estimating the premium-beta, one needs to specify the variables that are used to form expectations of the market premium each period
- That is, the information set that is behind γ_{1t-1}
- The predictability literature points out a number of variables that are able to predict the equity premium: dividend yields, term spread, default spread, etc.
- They focus on the default spread: the difference in yields between BAA and AAA rated bonds. Label it R_{t-1}^{prem}
- Also, they assume that the expected market risk premium is a linear function of this predictor

$$\gamma_{it-1} = \kappa_0 + \kappa_1 R_{t-1}^{prem}$$

- Then, by replacing the last definition into (4), one obtains:

$$E(R_{it}) = c_0 + c_m \beta_i + c_{prem} \beta_i^{prem} \quad (5)$$

where

$$\beta_i^{prem} = Cov(R_{it}, R_{t-1}^{prem}) / Var(R_{t-1}^{prem})$$

- Now β_i^{prem} can be easily estimated because it involves two observable returns

Total wealth portfolio

- The other issue they want to address is the definition of the total wealth portfolio
- Income from dividends represents only 3% of monthly personal income, whereas salaries represent 63%
- Total wealth portfolio needs to account for return on human capital
- They show that the return on human capital can be expressed as the growth rate of per capita labor income

$$R_t^{labor} = \frac{L_t - L_{t-1}}{L_{t-1}}$$

- Then, they assume that the return on the total wealth portfolio is a linear function of both the return on stocks and the return on human capital:

$$R_{mt} = \phi_0 + \phi_1 R_t^{vw} + \phi_2 R_t^{labor}$$

where R_t^{vw} is the return on the value-weighted stock market index

- As a consequence, the unconditional beta can be expressed as

$$\beta_i = b_{vw}\beta_i^{vw} + b_{labor}\beta_i^{labor}$$

where

$$\beta_i^{labor} = Cov(R_{it}, R_t^{labor}) / Var(R_t^{labor})$$

- So, that replacing β_i into (5) gets the final expression for the unconditional return as a three-beta model

$$E(R_{it}) = c_0 + c_{vw}\beta_i^{vw} + c_{prem}\beta_i^{prem} + c_{labor}\beta_i^{labor} \quad (6)$$

- This is their preferred empirical specification: the Premium-Labor (P-L) model

The AP tests

- They run two types of AP tests:

1. Cross-sectional Fama and MacBeth tests:

- They look at explanatory power (R^2) of model (6)
- They compare it with the explanatory power of a model that also includes size:

$$E(R_{it}) = c_0 + c_{vw}\beta_i^{vw} + c_{prem}\beta_i^{prem} \\ + c_{labor}\beta_i^{labor} + c_{size}\log(ME_{it})$$

where ME is the average market capitalization of portfolio i

- They also compute t-stats for the coefficients using Shanken (1992) correction

2. General Method of Moments (GMM) estimates of the model expressed in Stochastic Discount Factor (SDF) form

- This method requires weaker assumptions for computing the standard errors of the coefficients
- Also, it allows a comparison of different models on the basis of the Hansen-Jagannathan (H-J) distance

The GMM tests

- These tests are based on the moment conditions that the pricing errors of the model expressed in SDF form are zero
- In general one can always go from an SDF representation

$$\begin{aligned}E(R_{it}m_t) &= 1 \\m_t &= a - bf_t\end{aligned}$$

where m_t is the SDF and f_t is a K -vector, to an expected return-beta representation

$$\begin{aligned}E(R_{it}) &= \lambda_0 + \lambda'\beta \\ \beta &= Cov(R_{it}, f_t) E(f_t f_t)^{-1}\end{aligned}$$

and vice versa (see Cochrane ch. 6 and end of these lecture notes). This equivalence can be shown very easily by replacing the definition of m_t into the SDF representation

- In this particular case, the three factors that enter the SDF are R_t^{vw} , R_t^{prem} , R_t^{labor}

- So, the SDF representation is

$$E [R_{it}d_t(\delta)] = 1$$

where

$$d_t(\delta) = \delta_0 + \delta_{vw}R_t^{vw} + \delta_{prem}R_t^{prem} + \delta_{labor}R_t^{labor}$$

- Given that we have N assets, the N moment conditions for the vector of N pricing errors $w_t(\delta)$ are

$$E [w_t(\delta)] = 0$$

where

$$w_t(\delta) = R_t d_t(\delta) - \mathbf{1}_N$$

R_t is an N -vector of asset returns and $\mathbf{1}_N$ is an N -vector of ones

- The GMM estimates the 4 parameters in δ so as to minimize a quadratic form in the N moment conditions

$$E [w_t(\delta)]' A E [w_t(\delta)] \tag{7}$$

where A is a weighting matrix

- Assume that $w_t(\delta)$ is i.i.d. over time
- In this case, the optimal weighting matrix by Hansen and Singleton (1982) reduces to

$$A = [Var(w_t(\delta))]^{-1} \quad (8)$$

which is also called **second-stage** weighting matrix, because you estimate it by computing pricing errors from a first stage in which the weighting matrix is typically the identity

- However, the optimal quadratic form using the weighting matrix in (8) cannot be used as a metric to make comparisons across different asset pricing models
 - If a model contains more noise, $Var(w_t(\delta))$ is larger
 - Then the quadratic form is smaller just because of the noise, and not because of smaller pricing errors (Remember Fama and French's (1993) defense against the rejection of their model by the GRS test)
- The solution proposed by Hansen and Jagannathan (1994) is to use

$$A = [E(R_t R_t)]^{-1}$$

which is the matrix of second moments of returns

- The advantage is that this matrix remains the same across different specifications of the AP model and allows comparisons of different models
- Also, the authors show that the square root of the resulting quadratic form, called the *Hansen-Jagannathan (H-J) distance*, is the pricing error of the most mispriced portfolio among the N assets by a given AP model
- They derive the distribution of this statistic, which is in general non-standard

The Data

- Test assets: 100 size-beta sorted portfolios (Fama and French, 1992)
- Small stocks have higher returns which are not matched by their unconditional betas
- Do not condition on Compustat availability
- Avoid using B/M data because of survivorship bias
- Compustat backfills two years of data when adding a firm to the data set
- Hence, first two years of data are on average 'good' years because only surviving firms can be added
- Kothari, Shanken, and Sloan (1995) provide evidence of a 10% upward bias in average returns for small stocks
- Normally, one tries to avoid the bias by omitting the first two years of data
- Issue: J&W don't try to price the Value Premium, which is harder

Results: Unconditional CAPM

- The tested model is

$$E(R_{it}) = c_0 + c_{vw}\beta_i^{vw}$$

- Panel A of Table II:

Panel A: The Static CAPM without Human Capital						
Coefficient:	c_0	c_{vw}	c_{prem}	c_{labor}	c_{size}	R-square
Estimate:	1.24	-0.10				1.35
t-value:	5.17	-0.28				
p-value:	0.00	78.00				
Corrected-t:	5.16	-0.28				
Corrected-p:	0.00	78.01				
Estimate:	2.08	-0.32			-0.11	57.56
t-value:	5.79	-0.94			-2.30	
p-value:	0.00	34.54			2.14	
Corrected-t:	5.77	-0.94			-2.30	
Corrected-p:	0.00	34.60			2.17	
Coefficient:	δ_0	δ_{vw}	δ_{prem}	δ_{labor}		HJ-dist
Estimate:	0.97	1.55				0.6548
t-value:	89.01	1.09				
p-value:	0.00	27.59				0.22

- Fama and MacBeth Regressions:

- Negative coefficient on β^{vw} : -0.10 and insignificant
- R^2 is only 1.35%
- Size is statistically significant and increases R^2
- GMM estimates:
 - δ_{vw} is insignificant: the market proxy R_t^{vw} has no importance in SDF
 - the H-J distance is significantly different from zero (p-value = 0.22%)
- Overall: very poor performance of the model

Conditional CAPM without Human Capital

- The tested model allows for covariation of β_{it-1} with γ_{it-1}

$$E(R_{it}) = c_0 + c_{vw}\beta_i^{vw} + c_{prem}\beta_i^{prem}$$

- Panel B of Table II:

Panel B: The Conditional CAPM without Human Capital

Coefficient:	c_0	c_{vw}	c_{prem}	c_{labor}	c_{size}	R-square
Estimate:	0.81	-0.31	0.36			29.32
t-value:	2.72	-0.87	3.28			
p-value:	0.66	38.45	0.10			
Corrected-t:	2.19	-0.70	2.67			
Corrected-p:	2.87	48.43	0.75			
Estimate:	1.77	-0.38	0.16		-0.10	61.66
t-value:	4.75	-1.10	2.50		-1.93	
p-value:	0.00	27.17	1.26		5.35	
Corrected-t:	4.53	-1.05	2.40		-1.84	
Corrected-p:	0.00	29.53	1.66		6.59	

Coefficient:	δ_0	δ_{vw}	δ_{prem}	δ_{labor}	HJ-dist
Estimate:	1.48	2.05	-45.94		0.6425
t-value:	6.71	1.47	-2.36		
p-value:	0.00	14.14	1.83		0.98

- Fama and MacBeth regressions:

- β^{vw} keeps being insignificant
 - β^{prem} is significantly different from zero
 - R^2 increases substantially to 29.32%
 - Size is only marginally significant now. So, size seems to be proxying for covariation of β and expected risk premium
- GMM estimates:
 - δ_{prem} is significant
 - The H-J distance is still significantly different from zero: the model still performs poorly

Preferred specification: P-L model

- The model includes beta instability and human capital:

$$E(R_{it}) = c_0 + c_{vw}\beta_i^{vw} + c_{prem}\beta_i^{prem} + c_{labor}\beta_i^{labor}$$

- Panel C of Table II:

Panel C: The Conditional CAPM with Human Capital

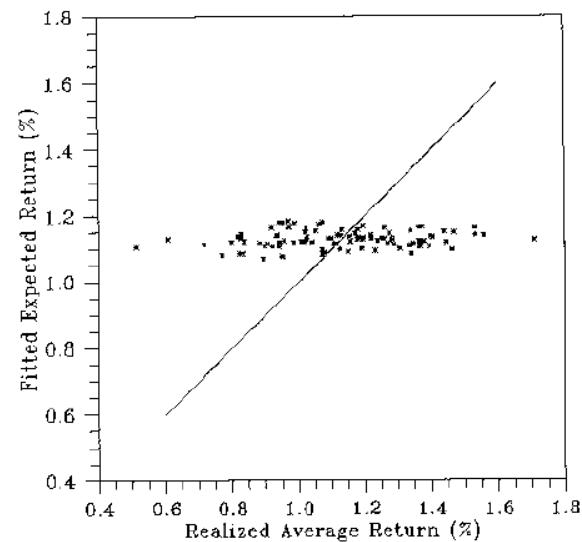
Coefficient:	c_0	c_{vw}	c_{prem}	c_{labor}	c_{size}	R-square
Estimate:	1.24	-0.40	0.34	0.22		55.21
t-value:	5.51	-1.18	3.31	2.31		
p-value:	0.00	23.76	0.09	2.07		
Corrected-t:	4.10	-0.88	2.48	1.73		
Corrected-p:	0.00	37.99	1.31	8.44		
Estimate:	1.70	-0.40	0.20	0.10	-0.07	64.73
t-value:	4.61	-1.18	3.00	2.09	-1.45	
p-value:	0.00	23.98	0.27	3.62	14.74	
Corrected-t:	4.14	-1.06	2.72	1.89	-1.30	
Corrected-p:	0.00	29.07	0.66	5.87	19.29	
Coefficient:	δ_0	δ_{vw}	δ_{prem}	δ_{labor}		HJ-dist
Estimate:	2.26	1.81	-65.72	-97.72		0.6184
t-value:	6.39	1.26	-3.10	-2.94		
p-value:	0.00	20.65	0.20	0.33		19.38

- Fama and MacBeth regressions:

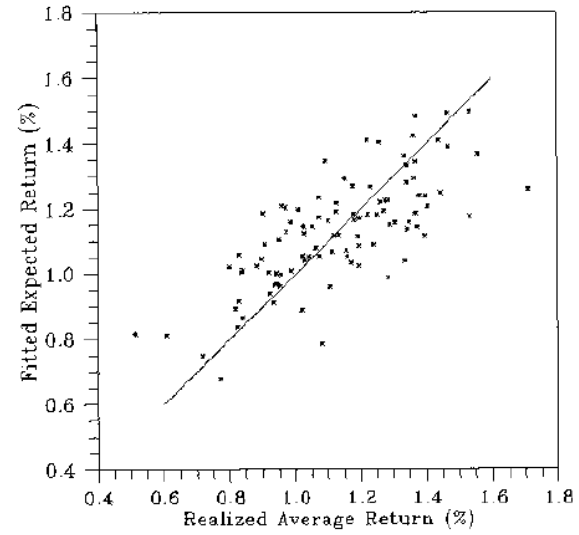
- R^2 increases to 55.21% (is it corrected R^2 ? It does not seem to be)
- c_{labor} is significant
- Size is no longer significant
- GMM estimates:
 - H-J drops and is no longer significant (p-value = 19.38%)
- Weak points:
 - β^{vw} is still not significant in explaining the cross-section of returns
 - The zero beta rate c_0 is still very high (1.24% monthly), higher than any plausible risk free rate, suggesting that there is still a part of expected returns that is left unexplained
 - Impact of the number of factors on test statistics (R^2 , HJ-distance)?

Graphical Inspection

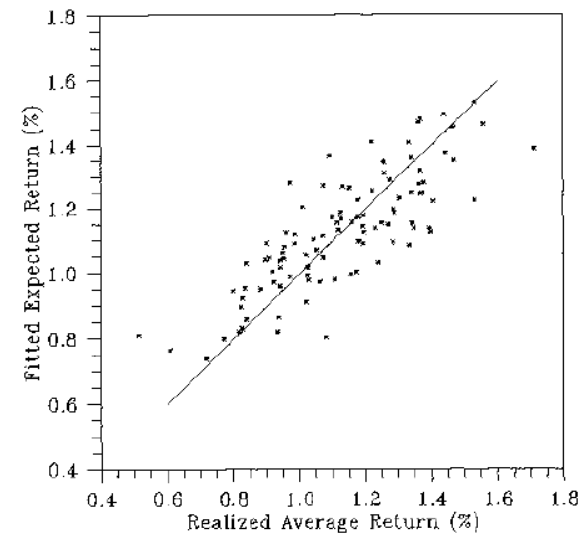
- It could be that improvement in statistical performance is due to a few outliers that can be better priced
- One would like general assessment of performance
- Look at how predicted expected return using estimated coefficients aligns with realized average returns
- If model works perfectly, should get a scatter plot aligned on the 45° line
- Unconditional CAPM:



- P-L model:



- P-L model with Size:



- The P-L model manages to improve the overall fit and size does not add much to the explanatory power

Comparison with F&F model

- Is the effect of the two additional factors introduced by F&F subsumed by the P-L model?
- Tested models are F&F:

$$E(R_{it}) = c_0 + c_{vw}\beta_i^{vw} + c_{smb}\beta_i^{smb} + c_{hml}\beta_i^{hml}$$

and horse-race of F&F and P-L model

$$E(R_{it}) = c_0 + c_{vw}\beta_i^{vw} + c_{prem}\beta_i^{prem} + c_{labor}\beta_i^{labor} \\ + c_{smb}\beta_i^{smb} + c_{hml}\beta_i^{hml}$$

- Results in Table IV:

Coefficient:	c_0	c_{vw}	c_{prem}	c_{labor}	c_{SMB}	c_{HML}	R-square
Estimate:	1.39	-0.45			0.33	0.25	55.12
t-value:	6.07	-0.95			1.53	0.96	
p-value:	0.00	34.34			12.60	33.59	
Corrected-t:	5.99	-0.94			1.51	0.95	
Corrected-p:	0.00	34.97			13.12	34.19	

Coefficient:	δ_0	δ_{vw}	δ_{prem}	δ_{labor}	δ_{SMB}	δ_{HML}	HJ-dist
Estimate:	0.98	2.62			-4.56	-0.94	0.6432
t-value:	35.00	1.35			-2.10	-0.28	
p-value:	0.00	17.78			3.60	77.91	0.65

Coefficient:	c_0	c_{vw}	c_{prem}	c_{labor}	c_{SMB}	c_{HML}	R-square
Estimate:	1.20	-0.38	0.22	0.11	0.16	0.22	64.04
t-value:	5.24	-0.80	3.32	2.25	0.78	0.84	
p-value:	0.00	42.41	0.09	2.44	43.79	40.24	
Corrected-t:	4.60	-0.70	2.95	1.99	0.68	0.74	
Corrected-p:	0.00	48.22	0.32	4.69	49.49	46.11	

Coefficient:	δ_0	δ_{vw}	δ_{prem}	δ_{labor}	δ_{SMB}	δ_{HML}	HJ-dist
Estimate:	2.17	2.62	-62.00	-89.33	-3.30	-0.59	0.6123
t-value:	6.09	1.26	-2.94	-2.67	-1.42	-0.18	
p-value:	0.00	20.90	0.32	0.77	15.52	85.98	18.58

- F&F model (top part):
 - Similar R^2 to P-L model
 - Similar zero-beta rate (high)
 - H-J distance is higher and significantly different from zero

- Horse-race (bottom part):
 - Neither of F&F factors are significant in GMM tests: it seems that P-L model subsumes their effect

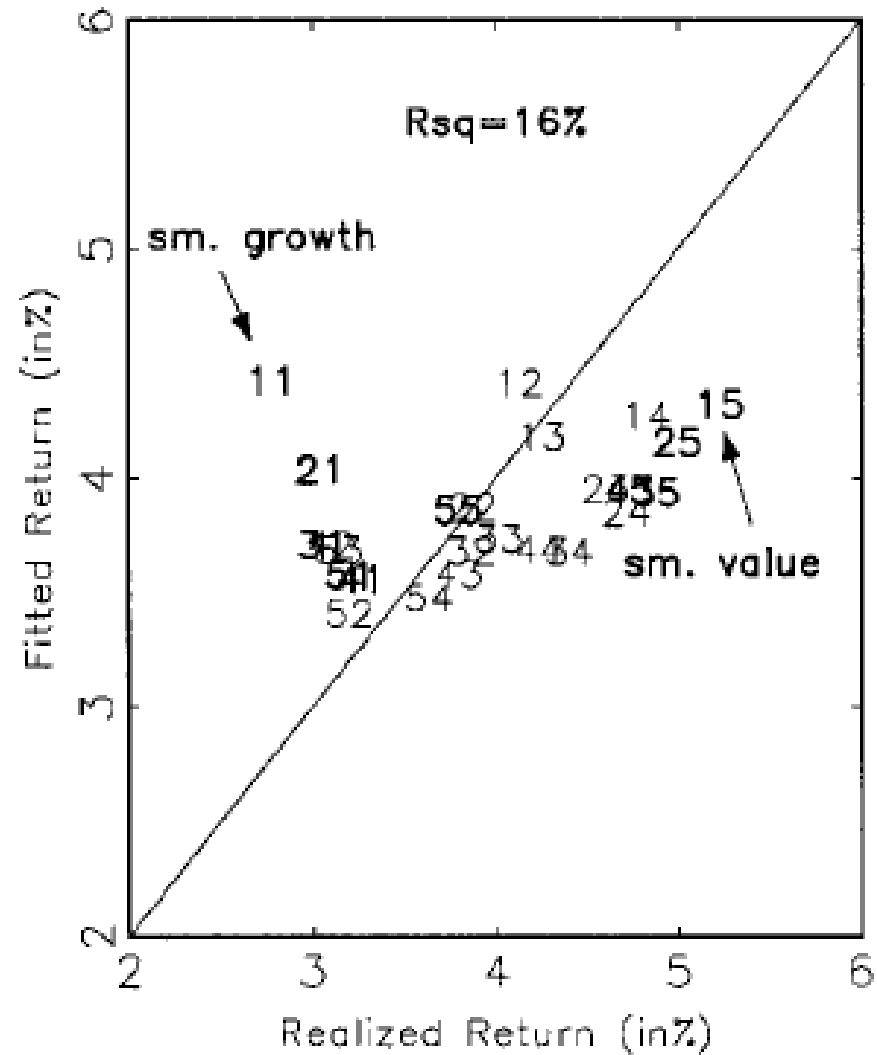
Conclusions

- It seems that including the instability risk and human capital covariation explains the higher return on small stocks
- Their story: small stocks have higher betas when the equity premium (γ_{1t}) is higher and their return covaries more strongly with human capital
- The paper does not provide evidence on performance in pricing the value premium, which is much higher than size effect
- Also, there is almost no comparison of estimated cross-sectional coefficients with the restrictions that the theory imposes on their values (they only discuss the zero-beta rate)
- For the last point, see Lewellen and Nagel (1996)

2. Lettau and Ludvigson, 2001

Motivation

- Failure of CAPM to explain size and B/M effects
- Success of F&F model, but why?
- Especially, failure of Consumption CAPM (CCAPM): riskiness of an assets derives from its covariance with consumption growth (see next graph)



- CCAPM is really appealing and it nests other models (CAPM, ICAPM) as special cases: consumption contains in itself all intertemporal considerations (hedging and speculative motives)

- Too bad it does not work!
- Is the failure of CCAPM due to the fact that it is estimated unconditionally?
- This paper is particularly interesting because of its textbook derivation of conditional AP models
- Also, it points out a cyclical variation in betas of value/growth stocks that is a promising direction for subsequent research
- The implementation of the tests is heavily criticized by Lewellen and Nagel

Conditional Asset Pricing (Cochrane ch. 8)

- In the case of the expected return-beta representation, we have already seen that a conditional model does not necessarily imply an unconditional model with the same factor
- Obviously, the same arguments hold for the SDF representation
- The coefficients in the SDF are time-varying
- Consider, for example, the SDF for CAPM

$$m_{t+1} = a_t + b_t R_{t+1}^m$$

- The pricing statement is also conditional on time t information

$$1 = E_t \left[\underbrace{\left(a_t + b_t R_{t+1}^m \right)}_{m_{t+1}} R_{it+1} \right]$$

- In general, this statement does not imply an unconditional pricing statement. Let's try. Take unconditional expectations of each side:

$$\begin{aligned}
 1 &= E \left[a_t R_{it+1} + b_t R_{t+1}^m R_{it+1} \right] \\
 &= E [a_t] E [R_{it+1}] + Cov (a_t, R_{it+1}) \\
 &\quad + E [b_t] E \left[R_{t+1}^m R_{it+1} \right] + Cov \left(b_t, R_{t+1}^m R_{it+1} \right)
 \end{aligned}$$

- Thus, you can get unconditional model only if the covariance terms are zero:

$$1 = E \left[\left(\underbrace{E [a_t]}_a + \underbrace{E [b_t]}_b R_{t+1}^m \right) R_{it+1} \right]$$

which is not in general the case

Hansen and Richard Critique

- Testing conditional models may not be enough to do things correctly
- To be sure to test the correct model, one needs to account for all the relevant conditioning information that investors use
- One needs to know investors' information set
- This is arguably impossible
- Hence, according to Hansen and Richard conditional factor models are not testable!
- It resonates with the Roll Critique

A partial solution: scaled factors

- Because investors' information changes over time, the parameters a_t and b_t in the SDF change
- One can try to model this time-variation by using a set of L instruments z_t , which are variables that plausibly enter investors' information set

$$\begin{aligned}a_t &= a'z_t \\ b_t &= b'z_t\end{aligned}$$

- Suppose, for example, 1 factor and 1 instrument

$$\begin{aligned}m_{t+1} &= a_t + b_t f_{t+1} \\ &= a_0 + a_1 z_t + (b_0 + b_1 z_t) f_{t+1} \\ &= a_0 + a_1 z_t + b_0 f_{t+1} + b_1 z_t f_{t+1}\end{aligned}$$

- In place of a conditional 1-factor model, one obtains an unconditional 3-factor model, with factors $(z_t, f_{t+1}, z_t f_{t+1})$ and fixed coefficients

- $z_t f_{t+1}$ are scaled factors
- In general, from K factors and L instruments one gets $(K + 1) \times (L + 1)$ unconditional factors (including a constant)
- This is a partial solution, because one could be omitting some relevant variable from z_t that is instead in investors' information set (Hansen and Richard Critique)

cay as an instrument

- In another paper, Lettau and Ludvigson (2001, JF) show that *cay*, the observable counterpart of the consumption-to-wealth ratio, works as a predictor of the equity premium
- Hence, they use *cay* as an instrument z_t
- The rationale comes from the intertemporal budget constraint:

$$W_{t+1} = (1 + R_{w,t+1})(W_t - C_t) \quad (9)$$

where W_t is total (human and non-human) wealth, and $R_{w,t+1}$ is the return on W_t

- Then, they log-linearize this constraint, iterate forward (see next slide), and take expectations, to get

$$c_t - w_t \approx E \left[\sum_{i=1}^{\infty} \rho_w^i (r_{w,t+i} - \Delta c_{t+i}) \right] \quad (10)$$

where small caps denote log variables, and r_w is the log return on total wealth

- The true consumption-to-wealth ratio is not observable because human capital is not observable
- For the present purposes, consider cay_t just as the observable counterpart of $c_t - w_t$
- It is evident from (10) that, as long as the expected growth rate of consumption is not too volatile, increases in the consumption-to-wealth ratio are related to expected increases in the return to non-human wealth (typically, stock market returns)
- Put simply: an increase in the consumption-to-wealth ratio relative to the average is possible if future returns on wealth are expected to be higher
- Hence, the consumption-to-wealth ratio rationally predicts stock returns, which is what they find empirically
- It then turned out that much of the predictability of cay resulted from in-sample fitting of a trend line (see “ tay ’s as good as cay ” by Brennan and Xia, 2005)

Log-linearization of the budget constraint

- Take the log of (9) and denote with small letters the log of a variable

$$w_{t+1} = r_{w,t+1} + \log(W_t - C_t)$$

Rearrange

$$r_{w,t+1} = w_{t+1} - \log \frac{W_t - C_t}{W_t} - \log W_t$$

$$r_{w,t+1} = w_{t+1} - w_t - \log \left(1 - \frac{C_t}{W_t} \right)$$

$$w_{t+1} - w_t = r_{w,t+1} + \log \left(1 - e^{c_t - w_t} \right)$$

Now, perform a linearization of the log around steady state values of c_t and w_t :

$$w_{t+1} - w_t = r_{w,t+1} + \log \left(1 - e^{c_0 - w_0} \right) - \frac{e^{c_0 - w_0}}{1 - e^{c_0 - w_0}} (c_t - w_t - (c_0 - w_0))$$

Let $\rho = 1 - e^{c_0 - w_0}$ and $k = \log(\rho) + \frac{1-\rho}{\rho} \log(1 - \rho)$ then you can write

$$\Delta w_{t+1} \approx r_{w,t+1} + k + (1 - 1/\rho)(c_t - w_t) \quad (11)$$

ρ can also be interpreted as the average ratio of invested wealth, $W - C$, to total wealth W . The lefthand side can be written as

$$\Delta w_{t+1} = \Delta c_{t+1} + (c_t - w_t) - (c_{t+1} - w_{t+1})$$

Sustitute this equation into (11)

$$c_t - w_t = \rho (r_{w,t+1} - \Delta c_{t+1}) + \rho (c_{t+1} - w_{t+1}) + \rho k$$

Now iterate this difference equation forward to infinity, impose the transversality condition

$$\lim_{t \rightarrow \infty} \rho^t (c_t - w_t) = 0$$

take expectations of both sides and obtain equation (10).

Why using a predictor of R^m as z_t ?

- Start again from the SDF representation and assume CAPM holds

$$1 = E_t \left[\underbrace{\left(a_t + b_t R_{t+1}^m \right)}_{m_{t+1}} R_{it+1} \right]$$

- You can obtain the following expected return-beta representation

$$E_t (R_{it+1}) = R_{f,t} - b_t R_{f,t} \text{Var}_t (R_{t+1}^m) \beta_{it}$$

(simply apply the SDF model to R_f to obtain the expression for a_t)

- Then, you can apply the model to price R_{t+1}^m itself and get

$$b_t = - \frac{E_t (R_{t+1}^m) - R_{f,t}}{R_{f,t} \text{Var}_t (R_{t+1}^m)}$$

- From this expression you see that an instrument for b_t needs to be a predictor of the equity premium. This explains why L&L choose $z_t = cay_t$

Testing the model

- They do the tests on the expected return-beta representation of the AP models with scaled factors
- Given $(K + 1) \times (L + 1)$ scaled factors in the unconditional SDF, you get an unconditional expected return-beta representation with betas on all these factors

$$E(R_{it+1}) = E(R_{f,t}) + \beta' \lambda$$

where β and λ are $[(K + 1) \times (L + 1) - 1]$ vectors

- They test different conditional and unconditional models
- The scaling variable is always *cay*
- Use Fama and MacBeth regressions
- They focus on R^2 as a measure of model performance (problem! see Lewellen, Nagel, and Shanken, 2007)
- Test assets: 25 F&F portfolios on quarterly data

Results for non-consumption models

- Table 1: tests of scaled and unscaled versions of: CAPM, Jag. and Wang; and unscaled version of F&F

ROW	CONSTANT	\widehat{cay}_t	Factors _{t+1}				$\widehat{cay}_t \cdot \text{Factors}_{t+1}$		R^2 (\bar{R}^2)
			R_{vw}	Δy	SMB	HML	R_{vw}	Δy	
1	4.18 (4.47) (4.45)		-.32 (-.27) (-.27)					.01 -.03	
2	3.21 (3.37) (1.87)		-1.41 (-1.20) (-.67)	1.26 (3.42) (1.90)				.58 .54	
3	1.87 (1.31) (1.21)		1.33 (.83) (.76)		.47 (.94) (.86)	1.46 (3.24) (2.98)		.80 .77	
4	3.70 (3.88) (2.61)	-.52 (-.22) (-.15)	-.06 (-.05) (-.03)				1.14 (3.59) (2.41)	.31 .21	
5	3.70 (3.86) (2.60)		-.08 (-.07) (-.44)				1.16 (3.58) (2.41)	.31 .25	
6	5.18 (5.59) (3.32)	-.44 (-1.60) (-.95)	-1.99 (-1.73) (-1.02)	.56 (2.12) (1.26)			.34 (1.67) (.99)	-.17 (-2.40) (-1.42)	.77 .71
7	3.81 (4.02) (2.80)		-2.22 (-1.88) (-1.31)	.59 (2.20) (1.53)			.63 (2.79) (1.94)	-.08 (-2.52) (-1.75)	.75 .70

- Uncond. CAPM: $\lambda < 0$, $R^2 = 0.01$

- Uncond. Jag. & Wang (HC): $R^2 = 0.58$
- F&F: $R^2 = 0.80$
- Scaled (or conditional) CAPM: $R^2 = 0.31$
- Scaled Jag & Wang: $R^2 = 0.77$
- Scaled factors are in general significant
- Conclusion: scaling improves performance and almost matches F&F
- Problem: zero-beta rate is too high (over 3% quarterly). Possible explanation: similar to Jag. & Wang, measurement error in macro scaling variables causes all coefficients to be biased towards zero. So, the intercept captures the unexplained part

Results for CCAPM

- Table 3: unscaled and scaled versions of CCAPM

Row	Constant	\widehat{cay}_t	Δc_{t+1}	$\widehat{cay}_t \cdot \Delta c_{t+1}$	R^2 (\bar{R}^2)
1	3.24		.22		.16
	(4.93)		(1.27)		.13
	(4.46)		(1.15)		
2	4.28	-.13	.02	.06	.70
	(6.10)	(-.43)	(.20)	(3.12)	.66
	(4.24)	(-.30)	(.14)	(2.17)	
3	4.10		-.02	.07	.69
	(6.82)		(-.14)	(3.20)	.66
	(5.14)		(-.10)	(2.41)	

- Unscaled (Row 1): $R^2 = 0.16$ (the graph we saw initially)
- Scaled (Rows 2 and 3): $R^2 = 0.70$
- Conclusions: CCAPM performs well when tested in conditional version (this fact explains the title of the paper)
- Problem: still zero beta rate very high

Intuition behind the result

- To obtain the betas for the second pass regression, one runs the time-series first pass regression:

$$R_{it+1} = \alpha^i + \beta_{\Delta c}^i \Delta c_{t+1} + \beta_{\Delta c \cdot z}^i \Delta c_{t+1} \cdot z_t + \beta_z^i z_t + u_{t+1}^i$$

for each asset, where $z_t = cay_t$

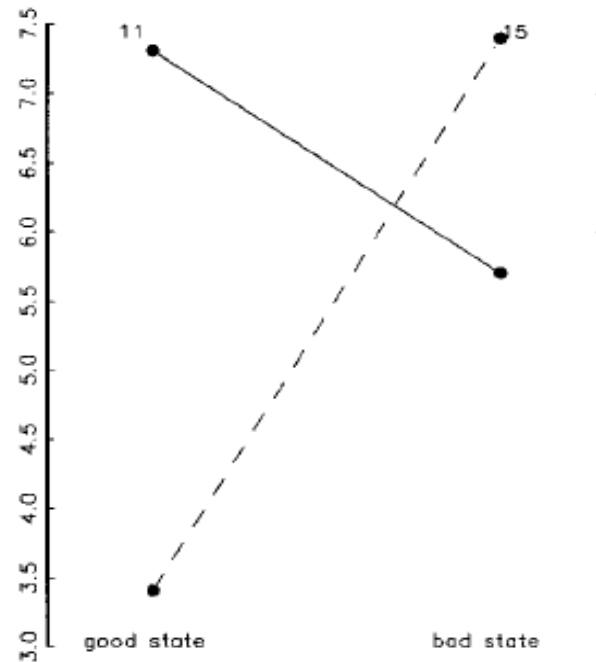
- Collecting terms in Δc_{t+1} one obtains

$$\begin{aligned} R_{it+1} &= \alpha^i + \beta_z^i z_t + B_t^i \Delta c_{t+1} + u_{t+1}^i \\ B_t^i &= \beta_{\Delta c}^i + \beta_{\Delta c \cdot z}^i z_t \end{aligned} \tag{12}$$

So, the conditional beta on consumption growth is time-varying as a function of cay

- Time-variation in B_t^i explains the risk premium
- They plot B_t^i for different assets in different states of the world: bad times (high cay , high risk premia, high risk aversion) and good times (low cay)

- Figure 2 (panel (a)):



- It turns out that small-value stocks (1,5) have higher betas in bad times. The opposite is true for small-growth (1,1) stocks.
- Because bad times are more infrequent than good times, the average beta of value stocks is low
- But they need to pay a high premium because they are the opposite of insurance: they perform poorly when investors are mostly risk averse (in bad states)

- The opposite holds for growth stocks

Other conditional CAPM papers

- After L&L a number of studies have looked for instruments to improve the performance of conditional AP models
- Lustig and Van Nieuwerburgh (2005, JF) use the ratio of housing wealth to human wealth (housing-collateral-ratio, my)
 - Idea: housing provides insurance (via its role as collateral) from labor income shocks
 - If housing value decreases, there is less collateral, less insurance, more exposure to shocks, more risk aversion, and higher risk premia are required to hold risky assets
 - Hence my predicts the equity premium
- Santos and Veronesi (2006, RFS) use the ratio of labor income to consumption
 - Idea: consumption is financed by both financial and non-financial (labor) income
 - Typically labor income plays a more important role than financial wealth

- If labor income decreases, the role of risky financial income goes up, investors care more about fluctuations in stock returns, and they require a higher premium to hold stocks
- Hence the ratio of labor income to consumption predicts the equity premium
- These papers and others manage to price the 25 size and B/M sorted portfolios of F&F

3. Lewellen and Nagel, 2006

Motivation

- Several recent papers (J&W, L&L, etc.) argue that conditional versions of CAPM and CCAPM solve asset pricing anomalies
- Question: is the covariation of β_t and γ_t empirically enough to generate the observed alphas in unconditional tests?
- Let us see the point. The conditional CAPM says (in J&W notation):

$$\begin{aligned}E_t \left(R_{it+1}^e \right) &= \beta_{it} \gamma_t \\ \beta_{it} &= \frac{\text{Cov}_t \left(R_{it+1}, R_{t+1}^m \right)}{\text{Var}_t \left(R_{t+1}^m \right)} \\ \gamma_t &= E_t \left(R_{t+1}^m \right) - R_{f,t}\end{aligned}$$

- Take unconditional expectations

$$\begin{aligned}E \left(R_{it+1}^e \right) &= E \left(\beta_{it} \gamma_t \right) \\ &= E \left(\beta_{it} \right) E \left(\gamma_t \right) + \text{Cov} \left(\beta_{it}, \gamma_t \right) \\ &= \bar{\beta}_i \gamma + \text{Cov} \left(\beta_{it}, \gamma_t \right)\end{aligned}\tag{13}$$

- By definition, the unconditional alpha in a time-series regression has the following Plim

$$\alpha_i^u = E(R_{it+1}^e) - \beta^u \gamma \quad (14)$$

where $\beta^u = \frac{Cov(R_{it+1}, R_{t+1}^m)}{Var(R_{t+1}^m)}$ is the unconditional beta

- Replacing (13) into (14) gets a clear expression for the unconditional alpha

$$\alpha^u = \gamma(\bar{\beta} - \beta^u) + Cov(\beta_t, \gamma_t)$$

- Under mild assumptions, you have that $\bar{\beta} = \beta^u$. So, that

$$\alpha^u = Cov(\beta_t, \gamma_t)$$

- L&L's point is that β_t varies along the business cycle with γ_t and this variation is enough to explain unconditional alphas
- L&N argue that this covariation is just not large enough. They show it in two ways:
 1. Calibrate $Cov(\beta_t, \gamma_t)$
 2. Compute conditional alphas directly and see that they are not zero

Calibration

- We can rewrite the unconditional alpha as

$$\alpha^u = Cov(\beta_t, \gamma_t) = \rho\sigma_\beta\sigma_\gamma$$

where ρ is the correlation between β_t and γ_t

- Notice that assets for which ρ is positive are predicted to have positive alphas
- Calibrate α^u by assuming values for $\rho, \sigma_\beta, \sigma_\gamma$:
 - $\sigma_\beta \in \{0.3, 0.5, 0.7\}$. For example, if $\beta = 1$ and $\sigma_\beta = 0.5 \implies$ a 95% confidence interval (CI) for β is $[0;2]$. F&F (1992) estimates of beta $\in [0.79;1.73]$. Hence, these assumptions produce a larger spread in beta than what is observed
 - $\sigma_\gamma \in [0.1\%; 0.5\%]$ monthly. For example, the average equity premium on 1964-2001 is about 0.47% monthly. Add $\pm 0.5\%$, you get 95% CI for equity premium is $[-6\%;18\%]$ annually. Hence, this assumption produces larger spreads than what observed and what is theoretically plausible

– $\rho \in \{0.6; 1\}$. No priors on this. Just postulate two values, where 1 is upper bound

● Results in Table 1:

$\rho = 0.6$	σ_β			$\rho = 1.0$	σ_β		
	0.3	0.5	0.7		0.3	0.5	0.7
	Unconditional alpha (%)				Unconditional alpha (%)		
$\sigma_\gamma = 0.1$	0.02	0.03	0.04	$\sigma_\gamma = 0.1$	0.03	0.05	0.07
0.2	0.04	0.06	0.08	0.2	0.06	0.10	0.14
0.3	0.05	0.09	0.12	0.3	0.09	0.15	0.21
0.4	0.07	0.12	0.17	0.4	0.12	0.20	0.28
0.5	0.09	0.15	0.21	0.5	0.15	0.25	0.35

- Calibrated alphas are small relative to empirical anomalies
- Max $\alpha^u = 0.35\%$ monthly, which is obtained with the most extreme assumptions on the three parameters
- In practice, Value – Growth (V–G) portfolio has $\hat{\alpha} = 0.59\%$, with $\hat{\sigma}_\beta = 0.25$. According to the table, its α^u should not be higher than 0.15%
- Momentum: Winners – Losers (W–L) portfolio has $\hat{\alpha} = 1\%$, with $\hat{\sigma}_\beta = 0.60$. According to the table, its α^u should not be higher than 0.35%

- The result of this exercise is that the covariance of β and γ is just too small to account for the observed anomalies (unless one is willing to accept implausible volatilities in betas and equity premium at monthly frequencies)

Direct test of Conditional CAPM

- Previous papers estimate the conditional CAPM by explicitly expressing the conditioning information and testing the unconditional model
- This is problematic if one does not observe all the variables in investors' information set
- L&N propose to estimate the conditional model directly by increasing the frequency of the data
- Assume that α_t and β_t are constant within a subperiod (e.g.: quarter, semester, year) and estimate the subperiod α_t and β_t using higher frequency data (e.g.: daily, weekly, monthly)
- For example: 21×3 daily returns are used to compute α_t and β_t for a given quarter. Or: 21×6 daily observations are used to compute the parameters in a semester
- There are microstructure issues (e.g. non-synchronous trading) in estimating high frequency parameters that they address

- The null is that the conditional CAPM holds in each subperiod:

$$H_0 : \alpha_t = 0 \quad \forall t$$

- You test it by forming time-series averages of the estimated α_t for each asset

$$\hat{\alpha} = \frac{1}{T} \sum_{t=1}^T \hat{\alpha}_t$$

like in Fama and MacBeth regressions. The standard error is the standard error of the mean

$$se(\hat{\alpha}) = \frac{1}{T^{1/2}} st.dev.(\hat{\alpha}_t)$$

- Crucial assumption: conditional beta does not move much within the subperiod

Results

- They look at B/M, Size, and Momentum portfolios: 1964-2001
- Conditional alphas in Table 3 (bold fonts denote significant estimates, estimates are in percent monthly):

	Size			B/M			Momentum		
	Small	Big	S-B	Grwth	Value	V-G	Losers	Winrs	W-L
<i>Average conditional alpha (%)</i>									
Quarterly	0.42	0.00	0.42	-0.01	0.49	0.50	-0.79	0.55	1.33
Semiannual 1	0.26	0.00	0.26	-0.08	0.40	0.47	-0.61	0.39	0.99
Semiannual 2	0.16	0.01	0.15	-0.12	0.36	0.48	-0.83	0.53	1.37
Annual	-0.06	0.08	-0.14	-0.20	0.32	0.53	-0.56	0.21	0.77

- Conditional alphas are very similar to unconditional alphas: large and significant for V-G, Small, and W-L
- They also look at how conditional betas covary with predicted values for the equity premium and show that the covariance is not large enough to explain the unconditional alphas
- Conclusion: conditional CAPM does not explain anomalies

Comparison with other studies

- Why then J&W, L&L, and others find that conditional CAPM works?
- L&N argue that these papers only test the *qualitative* implications of the AP models. They do not test the constraints imposed by the theory
- In other words, they treat some coefficients as free parameters, which allows them to get more explanatory power in the cross-sectional regressions

- Concretely, take the unconditional model

$$E(R_{it+1}^e) = \bar{\beta}_i \gamma + Cov(\beta_{it}, \gamma_t) \quad (15)$$

- Remember that L&L assume that conditional betas vary as a function of *cay* (see equation (12))
- Let's use L&N's notation in expressing this linear function

$$\beta_{it} = \beta_i + \delta_i cay_t \quad (16)$$

where δ_i and β_i are estimated in time-series regressions of $R_{i,t+1}$ on $R_{m,t+1}$ and $cay_t \times R_{m,t+1}$

- Replace (16) into (15)

$$E(R_{it+1}^e) = \bar{\beta}_i \gamma + \delta_i \text{Cov}(cay_t, \gamma_t) \quad (17)$$

- L&L test the asset pricing model in (17) by a regression of returns on $\bar{\beta}_i$ and δ_i

$$R_{it+1}^e = c_0 \bar{\beta}_i + c_1 \delta_i$$

treating c_1 as a free parameter, whereas one should compare it with $\text{Cov}(cay_t, \gamma_t)$

- L&L estimate $\hat{c}_1 = 0.06\%$ quarterly. Let's try to interpret it as an estimate of $\text{Cov}(cay_t, \gamma_t)$. As such it must be $< \sigma_\gamma \sigma_{cay}$

$$0.06\% < \sigma_\gamma \sigma_{cay}$$

with $\sigma_{cay} = 0.019$ estimated by L&L. So

$$\sigma_\gamma > \frac{0.06}{0.019}\% = 3.2\%$$

which means that the volatility of the equity premium is estimated to be higher than 3.2% quarterly

- This estimate seems implausible based on common sense (a 95% CI centered around zero would be [-25.6%;+25.6%] annually) and on calibration of theoretical models
- So, L&L's model would be rejected if fully tested against the restrictions imposed by the theory
- On a related note, the R^2 of these conditional models would drop substantially if the restrictions were imposed

Characteristics *are* Covariances

- Kelly, Pruitt, and Su (2019) revive the use of Principal Component Analysis (PCA) to identify latent factors and they combine it with conditional models to let loading on these latent factors be time-varying
- PCA identifies a set of factors that best explain the variance-covariance matrix of returns
- They call it Instrumented PCA (IPCA)
- The PCA literature started right after Ross's (1976) APT paper to identify the factors in the model (Chamberlain and Rothschild 1983; Connor and Korajczyk 1986, 1989)
- Problem with standard PCA: the identified latent factors did not have a clear economic interpretation and would not survive out-of-sample. Additionally, these models are static in that they do not allow for factor loadings to vary over time with characteristics
- The authors of the present paper make two considerations:

1. The standard way to proceed in asset pricing (à la Fama and French) assumes that factors are known, while in reality they are not. The estimation of latent factors gets around this problem
2. When a new characteristics seems to explain the cross-section of returns it is not typically tested against all previously known characteristics. This new approach allows to identify characteristics that eventually survive. It imposes a reduction of dimensionality in the number of characteristics)

Model Performance Assessment

- The model is estimated on the cross section of stocks, not portfolios: 12,000 stocks from 1962-2014
- Use R^2 from panel regressions of stock returns on the identified factors interacted with characteristics
- A high- R^2 in these regressions measures the performance of the model in explaining risk (or, we should say, *comovement*)
- Then, they use a cross-sectional R^2 in regressions of average returns on $\hat{\beta}_{it}\hat{\lambda}$ to measure the asset-pricing performance, that is the ability to explain the cross-section of returns
- Results:
 1. They focus on a model with five latent factors (adding more factors does not help)
 2. The time-series R^2 is 18.6%, close to the FF 5-factors' 21.9%

3. The cross-sectional R^2 is 0.7% low, but much better than FF's 0.3% (in sample)

4. Out of sample: 0.6% cross-sectional R^2 , still double the FF's R^2

- The model reduces dimensionality

$$r_{i,t+1} = z'_{i,t}\Gamma_a + z'_{i,t}\Gamma_\beta f_{t+1} + \varepsilon_{i,t+1}$$

- Instead of estimating a beta on each factor for each stock (about 60,000 estimates), it makes a beta depend on the stock characteristics via the matrix Γ_β
- With 5 factors, Γ_a is not significantly different from zero
- What do factors look like?
- F1 similar to leverage; F2 similar to the market; F3 similar to 12-month momentum; F4 similar to 1-month reversal; F5 is a mixed bag

Critical Assessment

- Lewellen and Nagel (2004) critique
- Comovement is not risk
- Lack of clear economic interpretation of factors

From SDF to $E(R)-\beta$

$$\begin{aligned} E(R_{it}m_t) &= 1 \\ m_t &= a - bf_t \end{aligned}$$

Replace m_t into the first equation and

$$\begin{aligned} E(R_{it}(a - bf_t)) &= 1 \\ aE(R_{it}) - bE(R_{it}f_t) &= 1 \\ aE(R_{it}) &= 1 + bE(R_{it}f_t) \\ aE(R_{it}) &= 1 + bE(R_{it})E(f_t) + bCov(R_{it}, f_t) \\ E(R_{it})(a - bE(f_t)) &= 1 + bCov(R_{it}, f_t) \end{aligned} \tag{18}$$

Note that by pricing the risk-free rate you get

$$\begin{aligned} R_f E(m) &= 1 \\ R_f &= \frac{1}{a - bE(f_t)} \end{aligned}$$

So, by replacing R_f into equation (18), you get

$$E(R_{it}) = R_f + \frac{b}{R_f} \text{Cov}(R_{it}, f_t)$$

which can be expressed in E(R)- β form

$$\begin{aligned} E(R_{it}) &= \lambda_0 + \beta_{if} \lambda_1 \\ \beta_{if} &= \frac{\text{Cov}(R_{it}, f_t)}{\text{Var}(f_t)} \\ \lambda_1 &= \frac{b}{R_f} \text{Var}(f_t) \\ \lambda_0 &= R_f \end{aligned}$$